Ballistics notes

Revision #213

Ben Trettel[1] January 29, 2019

Abstract

Models of pneumatic and spring gun ballistics are detailed. These models are applicable to potato and Nerf guns. Derivations are included as are references to other relevant papers and websites. Theory is compared against experimental data when possible. Limitations of the theory are made clear.

Disclaimer

This document is a work in progress. It was developed out of several different notes I wrote starting in 2009 and ending roughly in 2014. The notations are not entirely consistent, not everything is up to date, not everything is correct, and not everything is complete. This 2019 version has updated contact information but otherwise is unchanged aside from this disclaimer.

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1 Email: http://trettel.org/contact.html

1 Glossary

C:B ratio (CB) — the ratio of the chamber volume to barrel volume. Mathematically this is $\mathrm{CB} \equiv V_\mathrm{c}/V_\mathrm{b}$. This term came from spud-gun hobbyists [Spu08a]. In this document, the barrel volume is defined as the volume displaced by the projectile's trajectory, which does not include "dead space". Others might include dead space. This means that the alternative C:B ratio is $\widehat{\mathrm{CB}} \equiv V_\mathrm{c}/(V_\mathrm{b} + V_\mathrm{d}) = \mathrm{CB}\,/(1 + V_\mathrm{d}^*)$ where $V_\mathrm{d}^* \equiv V_\mathrm{d}/V_\mathrm{b} = V_\mathrm{d}/(A_\mathrm{b} L_\mathrm{b})$.

Dead space (V_d) — also called dead volume — the volume between the flow restriction in the system (whether a valve in a pneumatic or a contraction in a springer) and the projectile before firing.

Pressure of friction (P_f) — a way to describe friction on the projectile used originally by Hall [Hal08]. The normal force from the projectile pushing onto the inside of the barrel is rarely known (though it can be estimated). In a computer simulation, the only quantity you need is the actual resisting force of friction divided by the cross-sectional area of the barrel tube, which is essentially a pressure. Thus, $P_\mathrm{f} \equiv F_\mathrm{f}/A_\mathrm{b}$ where $F_\mathrm{f} \equiv \mu N$ where μ is the coefficient of friction and N is the normal force. The dynamic friction is denoted with P_fd and the static friction is denoted with P_fs.

Valve opening time (t_o) — the time it takes for the valve to go from completely closed to completely open. The completely open state may not be achieved in reality. The valve opening profile is the valve flow coefficient as a function of % open.

Dwell time (t_d) — the time from when the trigger is pulled to when the projectile leaves the barrel. The dwell time can be estimated from the muzzle velocity as $t_\mathrm{d} \approx L_\mathrm{b}/V_\mathrm{m}$.

Barrel length (L_b) — the length of the barrel. More specifically, the length from the back end of the projectile to the opposite end of the barrel. That is the choice assumed in my models.

Muzzle velocity (V_m) — the projectile's speed (so it's not a vector like velocity) as it exits the muzzle (end of the barrel). The incorrect usage of velocity is kept because this term is very common.

Flow factor (C^*) — a way to quantify how "fast" a valve opens. In pneumatic and spring guns with "fast" valves, the barrel and gas chamber or plunger pressures are both equal.

Fineness ratio (L/d) — also slenderness ratio or aspect ratio — sometimes seen as the variable δ —

Chambrage ($d_\mathrm{c}/d_\mathrm{b} = \sqrt{A_\mathrm{c}/A_\mathrm{b}}$) — the ratio of the chamber diameter to barrel diameter. Generally, a higher value of this quantity improves performance.

Beta factor (β) — the ratio of upstream pipe diameter to the minimum diameter of an orifice, e.g., $\beta \equiv d_\mathrm{u}/d_\mathrm{min}$. This is similar to the chambrage, though not the same aside from the case of pure reductions in area with no intermediate state.

Critical pressure ratio (b) — the pressure ratio which will "choke" the velocity - commonly denoted with b or $(P_\mathrm{d}/P_\mathrm{u})_\mathrm{crit}$. Choked flow occurs when the velocity in the valve/orifice/nozzle reaches a maximum and the velocity will not increase as the pressure ratio increases. However, note that mass flow rare can increase if upstream density increases. Many people (even authors of textbooks) seem to not understand that.

2 Internal ballistics and gun processes

2.1 Ideal barrel length

One of the holy grails of Nerf engineering is a rule for ideal or optimal barrel length. Below I detail how to estimate the ideal barrel length for the case where there are no flow restrictions and the plunger/chamber pressure is the same as the barrel pressure. This is usually not a realistic case, but it can help understanding of internal ballistics. More general and accurate equations for the ideal barrel length are detailed later in the internal ballistics section.

Approximate equations for the optimal C:B ratios can be found from adiabatic process relationships when C^* is high. C^* is dimensionless version of the flow coefficient between the gas chamber or plunger and the barrel. C^* is high for high valve/restriction flow rates (in this case, a valve/restriction that allows the barrel pressure to equal) and heavy darts. For a certain value of C^* (the "critical" value), the performance of a gun matches that of the adiabatic process. However, the transition criteria is a function of the static friction force on the dart in the barrel, along with potentially other things. The critical value decreases for static friction values that allow the pressure in the barrel to build up to or near the chamber pressure before the dart starts to move.

These solutions assume that the pressure in the gas chamber and barrel are identical and that the process occurs infinitesimally slow. Reversible adiabatic processes (which are isentropic processes) fit these conditions.

Numerical simulation results fit the pneumatic solution for high C^* and negligible V_d. The springer results have not been compared against numerical simulations.

Pneumatic case. This result is only valid for small V_d^* because in the initial state the gas in the dead space is neglected. Start with the adiabatic process relationship,

$$P_\mathrm{c} V_\mathrm{c}^\gamma = (P_\mathrm{atm} + P_\mathrm{f})(V_\mathrm{c} + V_\mathrm{b} + V_\mathrm{d})^\gamma \,, \tag{2.1}$$

and non-dimensionalize it to find the ideal C:B ratio equation:

$$\begin{aligned}
\frac{P_\mathrm{c}}{P_\mathrm{atm} + P_\mathrm{f}} &= \left(\frac{V_\mathrm{c} + V_\mathrm{b} + V_\mathrm{d}}{V_\mathrm{c}}\right)^\gamma \,, && (2.2)\\
&= \left(1 + \frac{V_\mathrm{b}}{V_\mathrm{c}} + \frac{V_\mathrm{d}}{V_\mathrm{b}}\frac{V_\mathrm{b}}{V_\mathrm{c}}\right)^\gamma \,, && (2.3)\\
\frac{P_\mathrm{c}/P_\mathrm{atm}}{(P_\mathrm{atm} + P_\mathrm{f})/P_\mathrm{atm}} &= \left[1 + \frac{1}{\mathrm{CB}}\left(1 + V_\mathrm{d}^*\right)\right]^\gamma \,, && (2.4)\\
\left(\frac{P_\mathrm{c}^*}{1 + P_\mathrm{f}^*}\right)^{1/\gamma} - 1 &= \frac{1}{\mathrm{CB}}(1 + V_\mathrm{d}^*) && (2.5)\\
\mathrm{CB} &= \frac{1 + V_\mathrm{d}^*}{\left(\frac{P_\mathrm{c}^*}{1 + P_\mathrm{f}^*}\right)^{1/\gamma} - 1} \,. && (2.6)
\end{aligned}$$

Then you can rearrange this to find the physical ideal length of the barrel,

$$\frac{V_\mathrm{c}}{\cancel{A_\mathrm{b} L_\mathrm{b}}} = \frac{1 + \dfrac{V_\mathrm{d}}{A_\mathrm{b} L_\mathrm{b}}}{\left(\dfrac{\dfrac{P_\mathrm{c}}{P_\mathrm{atm}}}{1 + \dfrac{P_\mathrm{f}}{P_\mathrm{atm}}} \right)^{1/\gamma} - 1} A_\mathrm{b} L_\mathrm{b} \,, \tag{2.7}$$

$$V_\mathrm{c} = \frac{A_\mathrm{b} L_\mathrm{b} + V_\mathrm{d}}{\left(\dfrac{P_\mathrm{c}}{P_\mathrm{atm} + P_\mathrm{f}} \right)^{1/\gamma} - 1} \,, \tag{2.8}$$

$$L_\mathrm{b} = \frac{V_\mathrm{c} \left[\left(\dfrac{P_\mathrm{c}}{P_\mathrm{atm} + P_\mathrm{f}} \right)^{1/\gamma} - 1 \right] - V_\mathrm{d}}{A_\mathrm{b}} \tag{2.9}$$

Spring case. The system starts at atmospheric pressure with the volume being the entire volume of the plunger tube (V_0) plus the dead space (V_d). The firing process ends with the displaced volume of the plunger replaced by the barrel volume (V_b), the dead space, and the final volume of the plunger tube (V_pf, i.e., the volume of the plunger tube when the dart leaves the barrel). In practice, the final volume of the plunger tube is not known and is sometimes assumed to be zero, though this is not rigorous.

Start, as before, with the adiabatic process relationship,

$$P_\mathrm{atm} (V_0 + V_\mathrm{d})^\gamma = (P_\mathrm{atm} + P_\mathrm{fd})(V_\mathrm{pf} + V_\mathrm{d} + V_\mathrm{b})^\gamma \,. \tag{2.10}$$

Following a procedure similar to the pneumatic case leads to

$$\mathrm{CB} = \frac{1 + V_\mathrm{d}^*}{(1 + V_\mathrm{d}^*) \left(\dfrac{1}{1 + P_\mathrm{fd}^*} \right)^{1/\gamma} - V_\mathrm{pf}^*} \,, \tag{2.11}$$

and

$$L_\mathrm{b} = \frac{(A_\mathrm{c} L_0 + V_\mathrm{d}) \left(\dfrac{P_\mathrm{atm}}{P_\mathrm{atm} + P_\mathrm{fd}} \right)^{1/\gamma} - V_\mathrm{pf} - V_\mathrm{d}}{A_\mathrm{b}} \,. \tag{2.12}$$

boltsniper's ideal barrel length. Readers might be aware of the results of some tests Evan Neblett (A.K.A. boltsniper) did in 2005 when he was designing the FAR [Neb06]. boltsniper, for the uninitiated, is one of the few engineers in the Nerf hobby, so his words have some authority behind them. People generally misrepresent what he said and overstate this formula's abilities. What boltsniper actually said is written below. The emphasis is mine.

> I did some experimentation to determine what would be the optimal barrel length for a given plunger size. *The goal was to find the barrel length for which the dart would exit the barrel as the plunger reaches the end of the plunger tube.* I started off by matching the volume of the plunger to the volume of the barrel. I knew that this was going to produce too long a barrel but it was a good place to start. This would assume that the air inside the plunger and barrel is incompressible and that there are no leaks. In the real world this is not the case. I reduced the barrel length until I had found the length at which the dart was leaving the barrel as the plunger was reaching its stop, coinciding with the maximum attainable range. Experimentally the plunger volume seems

> to be about 4 times that of the barrel. The relation for barrel to plunger size can be summed up in the following equation,
>
> $$4\pi r_{\mathrm{b}}^2 l_{\mathrm{b}} = \pi r_{\mathrm{p}}^2 l_{\mathrm{p}} \tag{2.13}$$
>
> where r_{b} is the barrel radius, r_{p} is the plunger radius, l_{b} is the barrel length, and l_{p} is the plunger length. For Nerf applications the barrel is almost always 1/2″ PVC or CPVC. r_{b} can then be set as a constant at 0.25″ and removed from the equation. Since we are trying to solve for the barrel length with a given plunger size, the equation can be rearranged and simplified to:
>
> $$l_{\mathrm{b}} = D_{\mathrm{p}}^2 l_{\mathrm{p}} \tag{2.14}$$
>
> This simple equation makes it easy to roughly but quickly size a barrel to a given plunger. The equation could also be used to size a plunger for a given length barrel. *This equation is based on experimental data and is not perfect. Four is not the golden number. This produces the optimal barrel length for the situation I was testing. The type of dart, dart-barrel friction, and total system volume will likely [affect] the optimal ratio. Nevertheless, the above equation can be used as a starting point.*

The last paragraph seems to be completely ignored by most people who use this formula. At best it's a starting point for further testing. The equation only applies to the FAR as that was all that he tested.

boltsniper later expanded on the restrictions on the use of this formula at NerfHaven [Neb05]:

> I derived that empirically and more importantly it was derived for the specific situation I intended on using it for: a plunger weapon. It will not work for a compressed air system. One of the big factors I used to come up with that was the lack of compressibility. I later factored that in with a constant that was derived empirically. *My tests were with a setup exactly like I was going to use on [the] finished product. If you scale the system down that magic constant may not hold true.*
>
> There are too many variables to analytically design the optimal barrel length. If you are going to build or mod a spring gun the equation I provided may be a good starting point. That equation gives a barrel length that is slightly too long, so to obtain the optimal length you are going to have to go shorter.
>
> The only real way to do it is experimentally.

Again, the emphasis is my own. The short message is that this equation only applies for the situation he was testing for.

But does it even apply for that situation? I'd argue no. boltsniper wasn't testing for optimal barrel length. In his own words (which I emphasized above), boltsniper's "goal was to find the barrel length for which the dart would exit the barrel as the plunger [reached] the end of the plunger tube." This does not coincide with when performance peaks based on my understanding of the interior ballistic processes.

Performance is maximized when acceleration slows to zero. If the plunger is at the end of the plunger tube, the pressure is likely near its peak. This corresponds to near peak acceleration because the force, not velocity, is near its peak. The ideal barrel length is probably longer in this case.

Also of interest is how he knew the plunger struck the end of the tube when the dart left. I seriously question how he determined that. The entire process occurs in a fraction of a second. He'd need a high speed camera with a clear plunger tube and barrel, some other optical system, some acoustic system, some magnetic system, or something I'm not considering to actually determine this with accuracy.

2.2 Energy efficiency

The energy efficiency of a Nerf gun is defined as the fraction of the input energy that is converted into muzzle kinetic energy of the dart:

$$\eta \equiv \frac{\text{muzzle kinetic energy}}{\text{input energy}} \,. \tag{2.15}$$

The muzzle energy is $T = m_\mathrm{d} V_\mathrm{m}^2/2$ where m_d is the dart mass and V_m is the muzzle velocity.

The best pneumatics rarely have energy efficiencies over 20%, while springers often have efficiencies of 35% or more. I suspect this is related to the unused gas left in the gas chamber of a pneumatic — a hybrid collapsing chamber gun (not unlike the current Nerf guns that use pneumatic cylinders) would combine the best of both worlds.

The efficiency of a small pneumatic gun I tested ranged from about 8% to 16% [Tre11b], while the efficiency of a modified Nerf Crossbow was about 39% [Tre09].

2.2.1 Overall energy efficiency

The overall energy efficiency is reduced by a combination of effects:

- inefficiencies of the internal ballistic process (η_i),
- energy lost due to heat losses for pneumatics (η_h),
- energy lost due to hysteresis for springers and pneumatics with latex tubing (η_s), and
- energy lost due to pump friction (η_p).

The combination of the last three inefficiencies can be thought of as a charging efficiency ($\eta_\mathrm{c} \equiv \eta_\mathrm{h}\eta_\mathrm{s}\eta_\mathrm{p}$).

2.2.2 Internal ballistic energy efficiency

The internal ballistic efficiency (η_i) is usually the least efficient part of the gun system.

Ideal gas pneumatics. Start with the work-energy principal and the adiabatic process law $PV^\gamma = \text{constant}$:

$$E = -\int_{V_1}^{V_2} P\, dV = P_1 V_1^\gamma \left(\frac{V_2^{1-\gamma} - V_1^{1-\gamma}}{\gamma - 1} \right) \,, \tag{2.16}$$

where E is the work into the system (i.e., the input energy).

I define $P_1 \equiv P_\mathrm{atm}$, $P_2 \equiv P_\mathrm{c,0}$, and $V_2 \equiv V_\mathrm{c}$ where γ is the ratio of specific heats (1.4 for air), P_atm is atmospheric pressure, $P_\mathrm{c,0}$ is the initial gas chamber pressure, and V_c is the gas chamber volume. I also define $P^*_\mathrm{c,0} \equiv P_\mathrm{c,0}/P_\mathrm{atm}$, the dimensionless initial gas chamber pressure. Using I can rewrite the input energy as

$$E = \frac{P_\mathrm{atm} V_\mathrm{c}}{\gamma - 1} \left[P^*_\mathrm{c,0} - (P^*_\mathrm{c,0})^{1/\gamma} \right] \tag{2.17}$$

after algebraic manipulation (best done with a computer to avoid mistakes).

The functional form of the energy efficiency for a pneumatic gun therefore is

$$\eta = \frac{(\gamma - 1) m_\mathrm{d} V_\mathrm{m}^2}{2 P_\mathrm{atm} V_\mathrm{c} \left[P^*_\mathrm{c,0} - (P^*_\mathrm{c,0})^{1/\gamma} \right]} \,. \tag{2.18}$$

For non-ideal gases and non-adiabatic compression processes, the math is similar.

Springers. The input energy is[1] $E = k(\Delta x)^2/2$ where k is the spring stiffness and Δx is the displacement of the spring. The functional form of the energy efficiency is therefore

$$\eta = \frac{m_\mathrm{d} V_\mathrm{m}^2/\cancel{2}}{k(\Delta x)^2/\cancel{2}} = \frac{m_\mathrm{d} V_\mathrm{m}^2}{k(\Delta x)^2} \, . \tag{2.19}$$

2.2.3 Heat loss efficiency

Gas reservoirs. The primary source of heat loss in a Nerf gun is from the gas reservoir. Due to adiabatic compression, the gas temperature in a gas reservoir will increase as the compression increases. For constant ratio of specific heats (γ) this entails

$$\frac{T_{\mathrm{c},0}}{T_\mathrm{atm}} = \left(\frac{P_{\mathrm{c},0}}{P_\mathrm{atm}}\right)^{(\gamma-1)/\gamma} \quad , \quad \frac{\rho_{\mathrm{c},0}}{\rho_\mathrm{atm}} = \left(\frac{P_{\mathrm{c},0}}{P_\mathrm{atm}}\right)^{1/\gamma} \quad . \tag{2.20}$$

Maximum energy loss. The gas reservoir will lose some total energy when the temperature is allowed to equilibrate. The equilibrium occurs when the gas temperature equals the atmospheric temperature. This is the worst-case scenario, and the associated efficiency, η_r, is easy to calculate.

As the equilibrium occurs, the gas mass density ρ will remain constant as no mass can leave the reservoir, in this model. More elaborate models could consider the effects of removing mass to fire the gun or perform other actions. Thus, $\rho_{\mathrm{r},0}$ (where r refers to the gas reservoir) is the same at the equilibrium state. We know the equilibrium temperature (T_atm), and thus by the ideal gas law we can calculate

$$P_{\mathrm{r},T_\mathrm{atm}} = \rho_\mathrm{atm} \left(\frac{P_{\mathrm{r},0}}{P_\mathrm{atm}}\right)^{1/\gamma} R T_\mathrm{atm} = P_\mathrm{atm} \left(\frac{P_{\mathrm{r},0}}{P_\mathrm{atm}}\right)^{1/\gamma} \quad . \tag{2.21}$$

If we define

$$\eta_\mathrm{r} \equiv \frac{E_{T_\mathrm{atm}}}{E_0} \, , \tag{2.22}$$

then we can use the definition of E_0 (eqn. 2.17) and a similar definition for E_{T_atm} (with $P = P_{\mathrm{r},T_\mathrm{atm}}$) to find η_r.

$$E_0 = \frac{P_\mathrm{atm} V_\mathrm{r}}{\gamma - 1} \left[P^*_{\mathrm{r},0} - (P^*_{\mathrm{r},0})^{1/\gamma} \right] \, , \tag{2.23}$$

$$E_{T_\mathrm{atm}} = \frac{P_\mathrm{atm} V_\mathrm{r}}{\gamma - 1} \left[P^*_{\mathrm{r},T_\mathrm{atm}} - (P^*_{\mathrm{r},T_\mathrm{atm}})^{1/\gamma} \right] \, . \tag{2.24}$$

After some manipulation of the equations above in the definition of η_r we find that

$$\eta_\mathrm{r} = \frac{{P^*_{\mathrm{r},0}}^{1/\gamma} - {P^*_{\mathrm{r},0}}^{1/\gamma^2}}{P^*_{\mathrm{r},0} - {P^*_{\mathrm{r},0}}^{1/\gamma}} \, . \tag{2.25}$$

Thus, the fraction of energy lost due to temperature equilibration of the gas reservoir is only a function of the initial pressures (and ratio of specific heats). This function is plotted below. Note that in the limit as $P^*_{\mathrm{r},0} \to 1$, $\eta_\mathrm{r} \to 1/\gamma$. This is the best equilibrated efficiency you can get. As a significant fraction of the energy can be lost, it is worthwhile to investigate how quickly this energy is lost for pneumatic guns with substantial air reservoirs.

1 Neglecting friction, only considering linear springs, and neglecting pre-stressed springs.

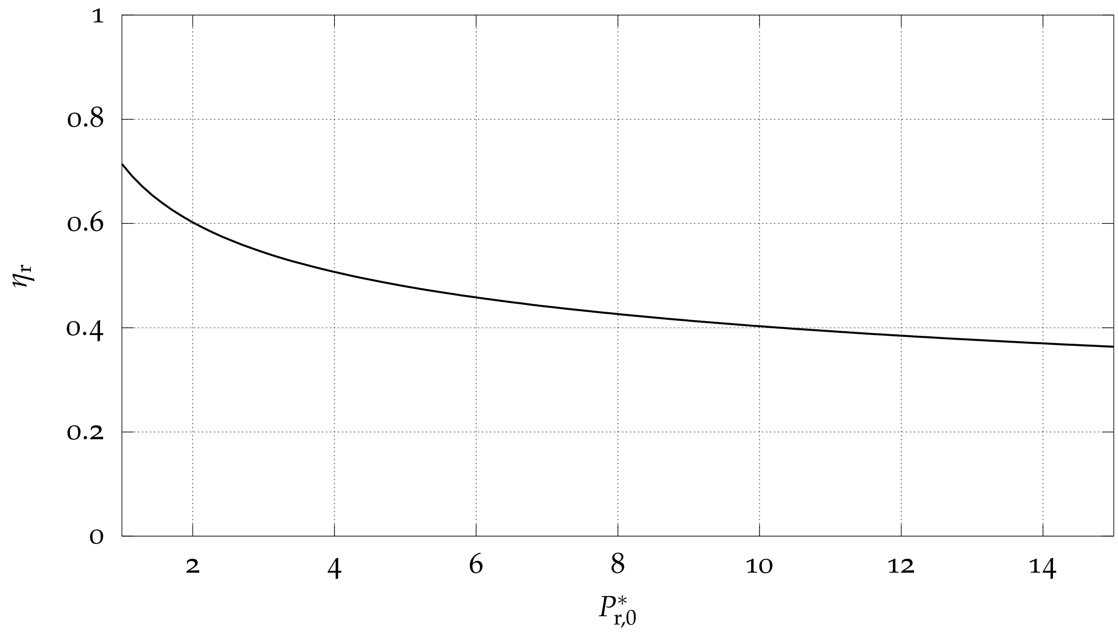

Figure 1: Worst-case-scenario gas reservoir energy efficiency (η_r) plotted as a function of dimensionless reservoir pressure, $P^*_{r,0} \equiv P_{r,0}/P_{atm}$.

Transient simulation. Assuming that the heat transfer occurs with negligible convection and that the air reservoir is a cylinder with no variation in the axial or θ directions allows us to use a 1D heat conduction equation [Inc+06, p. 74]:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(k\, r\frac{\partial T}{\partial r}\right) = \rho c_p \frac{\partial T}{\partial t}\ . \tag{2.26}$$

2.2.4 Pump friction efficiency

2.3 Valve and contraction models

2.3.1 Incompressible flow model

Start with Bernoulli's principle and the one-dimensional incompressible continuity equation ($A_1u_1 = A_2u_2$). Note that C_d here is the coefficient of discharge, not the coefficient of drag. Also note that $\beta^2 \equiv A_2/A_1$.

$$\tfrac{1}{2}\rho u_1^2 + P_1 = \tfrac{1}{2}\rho u_2^2 + P_2 \longrightarrow u_2 = \sqrt{\frac{2(P_2 - P_1)}{\rho(1-\beta^2)}} \tag{2.27}$$

$$Q = Au \tag{2.28}$$

$$SG \equiv \frac{\rho}{\rho_w} \tag{2.29}$$

$$Q = C_d A\sqrt{\frac{2\Delta P}{SG\rho_w(1-\beta^2)}} \tag{2.30}$$

A_2/A_1	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
C_c	0.617	0.624	0.632	0.643	0.659	0.681	0.712	0.755	0.813	0.892	1.00

Table 1: Experimental contraction coefficients for incompressible channel flows.

Most valve manufacturers give a flow coefficient, C_v^* (typically C_v, but C_v^* here to avoid confusion with the specific heat capacity at constant volume).

$$C_v^* = C_d A \sqrt{\frac{2}{\rho_w}} \longrightarrow Q = C_v^* \sqrt{\frac{\Delta P}{SG}} \tag{2.31}$$

$$C_d = \frac{C_v^*}{A} \sqrt{\frac{\rho_w}{2}} \tag{2.32}$$

2.3.2 Incompressible contraction flow data

Streeter [Str61, p. 3–21] lists contraction coefficients ($C_c \equiv A_c/A_2$) for incompressible channel? flows. A springer is an example of a nearly-incompressible contraction flow — the gas usually flows from the wide plunger tube to the thinner barrel.

Calvert [Cal03] is a good reference for the basics of contraction flows.

For orifices, similar data can be found in Merritt [Mer67, pp. 40–45].

2.3.3 Ideal nozzle flow model

Notes about the ideal nozzle flow equation:

- For valves and many orifices, the critical pressure ratio predicted by these equations is far too high [Bea10, p. 42].
- These equations were first derived by Saint-Venant and Wantzel [SW39].
- Equations like these are derived in Beater [Bea10], Blackburn, Reethof, and Shearer [BRS60], and Andersen [And67].

Velocity derivation. Euler's equation for the steady flow of an inviscid compressible flow is below [LR02].

$$\frac{u^2}{2} + \int \frac{dP}{\rho} = \text{constant} \tag{2.33}$$

Adiabatic process relationships for calorically perfect gases indicate the following is true:

$$\rho = \frac{P^{1/\gamma}}{C} \longrightarrow C = \frac{P_1^{1/\gamma}}{\rho_1} \tag{2.34}$$

$$\frac{u^2}{2} + C \int \frac{dP}{P^{1/\gamma}} = \text{constant} \longrightarrow \frac{u_2^2}{2} + \frac{P_1^{1/\gamma}}{\rho_1} \int_{P_1}^{P_2} \frac{dP}{P^{1/\gamma}} = \frac{u_1^2}{2} + \frac{P_1^{1/\gamma}}{\rho_1} \cancelto{0}{\int_{P_1}^{P_1} \frac{dP}{P^{1/\gamma}}} \tag{2.35}$$

$$\frac{u_2^2 - u_1^2}{2} = \frac{P_1^{1/\gamma}}{\rho_1} \int_{P_2}^{P_1} \frac{dP}{P^{1/\gamma}} \tag{2.36}$$

After performing the integration, we arrive at the following:

$$\frac{u_2^2 - u_1^2}{2} = \frac{P_1^{1/\gamma}}{\rho_1} \left(\gamma/(\gamma-1)\right) \left[P_1^{\frac{\gamma-1}{\gamma}} - P_2^{\frac{\gamma-1}{\gamma}}\right] \tag{2.37}$$

To find the mass flow rate through an orifice, nozzle, or valve, we need to find the area the gas passes through, the density of the gas there, and the velocity of the gas there (along with an empirical coefficient to account for non-idealities).

$$\dot{m} = C_d \rho_t A u_t \tag{2.38}$$

ρ_t is the density at the throat (thinnest portion) of the orifice, nozzle, or valve. $\rho_t = \rho_u \left(\frac{P_d}{P_u}\right)^{1/\gamma}$ by adiabatic process relationships.

Assuming that $u_1 = u_u = 0$, $u_t = u_2$, $P_1 = P_u$, and $P_2 = P_d$ we can rewrite the velocity equation.

$$\frac{u_t^2}{2} = \frac{P_u^{1/\gamma}}{\rho_u} \left(\gamma/(\gamma-1)\right) \left[P_u^{\frac{\gamma-1}{\gamma}} - P_d^{\frac{\gamma-1}{\gamma}}\right] \tag{2.39}$$

$$u_t = \sqrt{\frac{P_u^{1/\gamma}}{\rho_u} \left(\frac{2\gamma}{\gamma-1}\right) \left[P_u^{\frac{\gamma-1}{\gamma}} - P_d^{\frac{\gamma-1}{\gamma}}\right]} \tag{2.40}$$

Critical pressure ratio equation.

$$u_t = \sqrt{\frac{P_u^{1/\gamma}}{\rho_u} \left(\frac{2\gamma}{\gamma-1}\right) \left[\frac{\cancel{P_u^{\frac{\gamma-1}{\gamma}}}}{\cancel{P_u^{\frac{\gamma-1}{\gamma}}}} - \frac{P_d^{\frac{\gamma-1}{\gamma}}}{P_u^{\frac{\gamma-1}{\gamma}}}\right] P_u^{\frac{\gamma-1}{\gamma}}} \tag{2.41}$$

$$\rho_u = \frac{P_u}{RT_u} \tag{2.42}$$

$$u_t = \sqrt{RT_u \left(\frac{2\gamma}{\gamma-1}\right) \left[1 - \left(\frac{P_d}{P_u}\right)^{\frac{\gamma-1}{\gamma}}\right]} \tag{2.43}$$

$$c_t = \sqrt{\gamma R T_t} \tag{2.44}$$

$$M_t = \frac{u_t}{c_t} = \sqrt{\frac{T_u}{T_t} \left(\frac{2}{\gamma-1}\right) \left[1 - \left(\frac{P_d}{P_u}\right)^{\frac{\gamma-1}{\gamma}}\right]} \tag{2.45}$$

$$\frac{T_u}{T_t} = \left(\frac{P_u}{P_d}\right)^{\frac{\gamma-1}{\gamma}} \tag{2.46}$$

$$M_t = \frac{u_t}{c_t} = \sqrt{\left(\frac{2}{\gamma-1}\right) \left[\left(\frac{P_u}{P_d}\right)^{\frac{\gamma-1}{\gamma}} - 1\right]} \tag{2.47}$$

The authors assume that the velocity "chokes" when the throat Mach number equals 1. This can lead to a simple equation for the critical pressure ratio.

$$1 = \left(\frac{2}{\gamma - 1}\right)\left[\left(\frac{P_\mathrm{u}}{P_\mathrm{d}}\right)_\mathrm{crit}^{\frac{\gamma-1}{\gamma}} - 1\right] \tag{2.48}$$

$$\frac{\gamma - 1}{2} = \left(\frac{P_\mathrm{u}}{P_\mathrm{d}}\right)_\mathrm{crit}^{\frac{\gamma-1}{\gamma}} - 1 \tag{2.49}$$

$$\left(\frac{P_\mathrm{u}}{P_\mathrm{d}}\right)_\mathrm{crit}^{\frac{\gamma-1}{\gamma}} = 1 + \frac{\gamma - 1}{2} = \frac{\gamma + 1}{2} \tag{2.50}$$

$$a = \left(\frac{P_\mathrm{d}}{P_\mathrm{u}}\right)_\mathrm{crit} \tag{2.51}$$

$$b = \gamma/(\gamma - 1) \tag{2.52}$$

$$c = \frac{2}{\gamma + 1} \tag{2.53}$$

$$\frac{1}{a^{1/b}} = \frac{1}{c} \longrightarrow a^{1/b} = c \longrightarrow \left(a^{1/b}\right)^b = c^b \longrightarrow a = c^b \tag{2.54}$$

$$\left(\frac{P_\mathrm{d}}{P_\mathrm{u}}\right)_\mathrm{crit} = \left(\frac{2}{\gamma + 1}\right)^{\gamma/(\gamma-1)} . \tag{2.55}$$

The throat velocity is at the speed of sound whenever $\frac{P_\mathrm{d}}{P_\mathrm{u}} < b \equiv \left(\frac{P_\mathrm{d}}{P_\mathrm{u}}\right)_\mathrm{crit} \equiv \left(\frac{2}{\gamma + 1}\right)^{\gamma/(\gamma-1)}$. For air $b = 0.528$.

Unchoked mass flow rate equation. The parts of the mass flow rate equation can be combined to make an equation for the mass flow rate.

$$\dot{m} = C_\mathrm{d}\rho_\mathrm{u}\left(\frac{P_\mathrm{d}}{P_\mathrm{u}}\right)^{1/\gamma} A\sqrt{\frac{P_\mathrm{u}^{1/\gamma}}{\rho_\mathrm{u}}\left(\frac{2\gamma}{\gamma - 1}\right)\left[P_\mathrm{u}^{\frac{\gamma-1}{\gamma}} - P_\mathrm{d}^{\frac{\gamma-1}{\gamma}}\right]} \tag{2.56}$$

$$\dot{m} = C_\mathrm{d}A\left(\frac{P_\mathrm{d}}{P_\mathrm{u}}\right)^{1/\gamma}\sqrt{P_\mathrm{u}^{1/\gamma}\rho_\mathrm{u}\left(\frac{2\gamma}{\gamma - 1}\right)\left[\frac{\cancel{P_\mathrm{u}^{\frac{\gamma-1}{\gamma}}}}{\cancel{P_\mathrm{u}^{\frac{\gamma-1}{\gamma}}}} - \frac{P_\mathrm{d}^{\frac{\gamma-1}{\gamma}}}{P_\mathrm{u}^{\frac{\gamma-1}{\gamma}}}\right]P_\mathrm{u}^{\frac{\gamma-1}{\gamma}}} \tag{2.57}$$

$$\rho_\mathrm{u} = \frac{P_\mathrm{u}}{RT_\mathrm{u}} \tag{2.58}$$

$$\dot{m} = C_\mathrm{d}A\left(\frac{P_\mathrm{d}}{P_\mathrm{u}}\right)^{1/\gamma}\sqrt{\frac{P_\mathrm{u}^2}{RT_\mathrm{u}}\left(\frac{2\gamma}{\gamma - 1}\right)\left[1 - \left(\frac{P_\mathrm{d}}{P_u}\right)^{\frac{\gamma-1}{\gamma}}\right]} \tag{2.59}$$

$$\dot{m} = C_\mathrm{d}AP_\mathrm{u}\left(\frac{P_\mathrm{d}}{P_\mathrm{u}}\right)^{1/\gamma}\sqrt{\frac{2\gamma}{RT_\mathrm{u}(\gamma - 1)}\left[1 - \left(\frac{P_\mathrm{d}}{P_\mathrm{u}}\right)^{\frac{\gamma-1}{\gamma}}\right]} \tag{2.60}$$

Choked mass flow rate equation. $\dot{m} = C_\mathrm{d}\rho_\mathrm{t}Au_\mathrm{t}$ still applies. But $u_\mathrm{t} = c_\mathrm{t} = \sqrt{\gamma RT_\mathrm{t}}$ now.

$$\dot{m} = C_\mathrm{d} A \rho_\mathrm{u} \left(\frac{P_\mathrm{d}}{P_\mathrm{u}}\right)^{1/\gamma} \sqrt{\gamma R T_\mathrm{t}} \tag{2.61}$$

$$\frac{T_\mathrm{t}}{T_\mathrm{u}} = \left(\frac{P_\mathrm{d}}{P_\mathrm{u}}\right)^{\frac{\gamma-1}{\gamma}} = \left(\left(\frac{2}{\gamma+1}\right)^{\gamma/(\gamma-1)}\right)^{\frac{\gamma-1}{\gamma}} = \frac{2}{\gamma+1} \tag{2.62}$$

$$\dot{m} = C_\mathrm{d} A \rho_\mathrm{u} \left(\left(\frac{2}{\gamma+1}\right)^{\gamma/(\gamma-1)}\right)^{1/\gamma} \sqrt{\frac{2\gamma R T_\mathrm{u}}{\gamma+1}} \tag{2.63}$$

$$\rho_\mathrm{u} = \frac{P_\mathrm{u}}{R T_\mathrm{u}} \tag{2.64}$$

$$\dot{m} = C_\mathrm{d} A \frac{P_\mathrm{u}}{R T_\mathrm{u}} \left(\frac{2}{\gamma+1}\right)^{\frac{1}{\gamma-1}} \sqrt{\frac{2\gamma R T_\mathrm{u}}{\gamma+1}} = C_\mathrm{d} A P_\mathrm{u} \left(\frac{2}{\gamma+1}\right)^{\frac{1}{\gamma-1}} \sqrt{\frac{\gamma}{R T_\mathrm{u}} \frac{2}{\gamma+1}} \tag{2.65}$$

$$\dot{m} = C_\mathrm{d} A P_\mathrm{u} \left(\frac{2}{\gamma+1}\right)^{\frac{1}{\gamma-1}+\frac{1}{2}} \sqrt{\frac{\gamma}{R T_\mathrm{u}}} = C_\mathrm{d} A P_\mathrm{u} \left(\frac{2}{\gamma+1}\right)^{\frac{2+\gamma-1}{2(\gamma-1)}} \sqrt{\frac{\gamma}{R T_\mathrm{u}}} \tag{2.66}$$

$$\dot{m} = C_\mathrm{d} A P_\mathrm{u} \sqrt{\frac{\gamma}{R T_\mathrm{u}} \left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{\gamma-1}}} = C_\mathrm{d} A P_\mathrm{u} \sqrt{\frac{\gamma}{R T_\mathrm{u} \left(\frac{\gamma+1}{2}\right)^{\frac{\gamma+1}{\gamma-1}}}} \tag{2.67}$$

2.3.4 Pneumatic Drives modification of Sanville/ISO-6358 flow model [Bea10, p. 46]

$$\dot{m} = \begin{cases} 1000 C \rho_\mathrm{ref} \sqrt{1 - \left(\frac{b_\mathrm{l} - b}{1 - b}\right)^2} & \text{if } \frac{P_\mathrm{d}}{P_\mathrm{u}} > b_\mathrm{l} \text{ (laminar)} \\ P_\mathrm{u} C \rho_\mathrm{ref} \sqrt{\frac{T_\mathrm{ref}}{T_\mathrm{u}}} \sqrt{1 - \left(\frac{P_\mathrm{d}/P_\mathrm{u} - b}{1 - b}\right)^2} & \text{if } b_\mathrm{l} > \frac{P_\mathrm{d}}{P_\mathrm{u}} > b \text{ (subsonic)} \\ P_\mathrm{u} C \rho_\mathrm{ref} \sqrt{\frac{T_\mathrm{ref}}{T_\mathrm{u}}} & \text{if } b \geq \frac{P_\mathrm{d}}{P_\mathrm{u}} \text{ (sonic)} \end{cases} \tag{2.68}$$

Beater suggests using $b_\mathrm{l} \equiv 0.999$ as an arbitrary selection. A more physical method would use the Reynolds number as the distinguishing factor for the flow equations. The reference values are

$$\rho_\mathrm{ref} = 1.185\,\mathrm{kg/m^3} \quad , \quad T_\mathrm{ref} = 293.15\,\mathrm{K}\,. \tag{2.69}$$

2.3.5 Critical pressure ratio b

The critical pressure ratio equation in the nozzle flow section (eqn. 2.55) is often inaccurate. One explanation of the inaccuracy is that the maximum flow velocity is sonic, but the average velocity through the valve is lower. The nozzle flow model does not account for the velocity profile, so this inaccuracy can not be avoided. Beater [Bea10] notes that the critical pressure ratio typically has a value below 0.5. Valves generally are above 0.2. Some components can have a critical pressure ratio as low as 0. Some valve data sheets list critical pressure ratios. See Norgen [Nor96] for an example for QEVs (listed values are 0.5, 0.45 and 0.4).

Beater [Bea10, p. 48] cites Gidlund [Gid77], who gives the following equation for orifices with a length less than 10 times the diameter of the orifice:

$$b = 0.41 + 0.272\sqrt{\beta} \quad , \quad \beta \equiv \frac{d}{D} \tag{2.70}$$

where

d is the diameter of the orifice, and

D is the diameter of the pipe.

Shannak [Sha02] suggests the following relationship for orifices:

$$b = \left[0.014\beta^2 - 0.005\beta + 0.04\right] \gamma^4 \, . \tag{2.71}$$

That expression is approximately constant for all values of β. Shannak suggests the following approximation:

$$b = 0.041\gamma^4 \, . \tag{2.72}$$

2.3.6 Valve opening

Valves never open instantaneously. A model of the opening process could scale the flow coefficient some way in time.

2.3.7 Forces on interior of valves

2.3.8 Converting between flow coefficients

2.3.9 Common valve flow units

2.4 Pneumatic gun ballistics

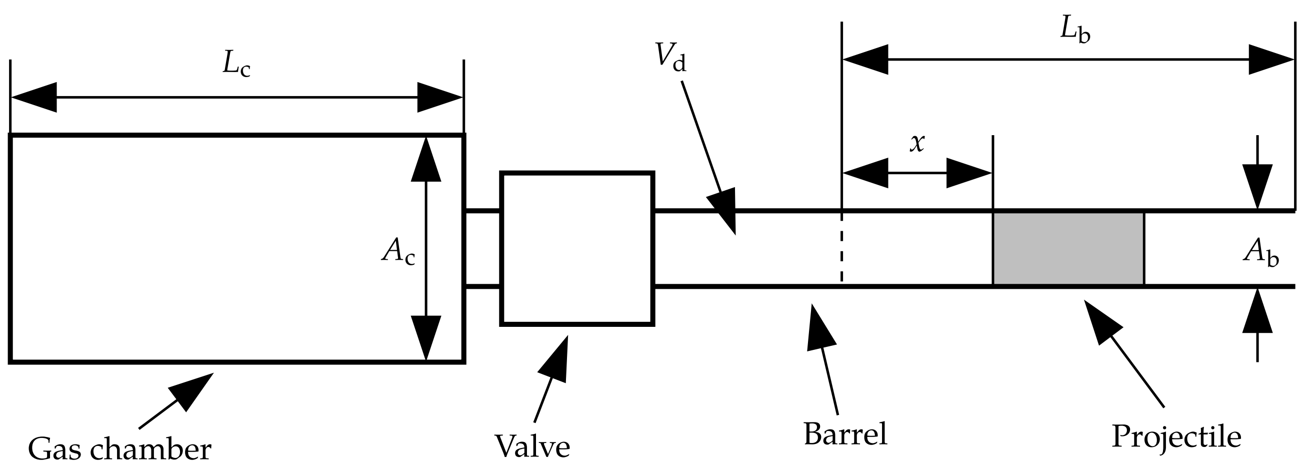

Figure 2: Some geometric variables and components of the model pneumatic gun.

2.4.1 Model pneumatic gun setup

The basic setup for the model gun is shown in figure 2. This pneumatic gun has a pressurized gas chamber, a valve, a barrel, and a projectile. The gas chamber, valve, and barrel are connected in series.

2.4.2 Subscripts

The subscript c refers to the gas chamber. The subscript b refers to the barrel. The subscript atm and a refers to atmospheric conditions.

2.4.3 Variables and constants

Variables		
F_d	$[F]$	Force on the projectile from barrel gas
$\dot{m}$	$[M][T]^{-1}$	Valve mass flow rate
P_c	$[F][L]^{-2}$	Chamber gas pressure
P_b	$[F][L]^{-2}$	Barrel gas pressure
ρ_b	$[M][L]^{-3}$	Barrel gas density
ρ_c	$[M][L]^{-3}$	Gas chamber gas density
T_b	$[\Theta]$	Barrel gas temperature
T_c	$[\Theta]$	Chamber gas temperature
T_u	$[\Theta]$	Temperature upstream of the valve
x	$[L]$	Projectile position referenced against starting position
$\dot{x}$	$[L][T]^{-1}$	Projectile velocity
$\ddot{x}$	$[L][T]^{-2}$	Projectile acceleration

Constants		
A_b	$[L]^2$	Barrel internal cross-sectional area
A_c	$[L]^2$	Chamber internal cross-sectional area
c_p	$[L]^2[T]^{-2}[\Theta]^{-1}$	Specific heat at constant pressure
c_v	$[L]^2[T]^{-2}[\Theta]^{-1}$	Specific heat at constant volume
L_b	$[L]$	Barrel length
L_c	$[L]$	Chamber length
m_d	$[M]$	Dart mass
P_atm	$[F][L]^{-2}$	Atmospheric pressure
P_f	$[F][L]^{-2}$	Pressure of friction
R	$[L]^2[T]^{-2}[\Theta]^{-1}$	Specific gas constant
T_atm	$[\Theta]$	Atmospheric temperature
V_c	$[L]^3$	Chamber volume ($V_\mathrm{c} \equiv A_\mathrm{c} L_\mathrm{c}$)
V_d	$[L]^3$	Dead volume

All pressures are absolute. The "pressure of friction" is the total friction force on the projectile divided by the barrel cross-sectional area, A_b. If you can blow a dart down a barrel with your lungs alone, then you know the upper limit of the pressure of friction is your maximum lung pressure, which is about 1.4 psi [Dea03].

The specific gas constant is not the universal gas constant ($\overline{R} = 8.3145\,\mathrm{J/mol/K}$). The specific gas constant is defined $R \equiv \overline{R}/M$ where M is the molecular weight.

2.4.4 Equations governing pnuematic guns

Non-dimensionalization

2.4.5 Analytical lumped-parameter model for pneumatics

Major assumptions and simplifications.

- Ideal gas with constant specific heats (a "calorically perfect gas").
- Velocities involves are "slow".
  - Can treat some stagnation temperatures and pressures as the normal temperatures and pressures.
  - Lumped parameter assumption for pressure, density, and temperature.
- The effect of the valve can be represented as a constant ratio between the barrel and gas chamber pressures.
- Frictionless flow (but friction on the projectile is included)
- Valve is assumed to not store any gas (has zero volume)
- No leaks are present (including leaks around the projectile)
- Air cylinders or other collapsing chambers are not part of the model.
- Phase changes are not part of this model.
- Elevation changes are negligible.

Model derivation. The following equations govern the performance of low-speed pneumatic guns:

$$V_\mathrm{c}\dot{\rho}_\mathrm{c} = -\dot{m}\,\mathrm{sgn}(P_\mathrm{c} - P_\mathrm{b})\,, \tag{2.73}$$

$$V_\mathrm{c}\frac{d}{dt}\left(\rho_\mathrm{c} c_\mathrm{v} T_\mathrm{c}\right) = -\dot{m} c_\mathrm{p} T_\mathrm{u}\,\mathrm{sgn}(P_\mathrm{c} - P_\mathrm{b})\,, \tag{2.74}$$

$$\dot{m} = ?\,, \tag{2.75}$$

$$\frac{d}{dt}\left(\rho_\mathrm{b}[V_\mathrm{d} + A_\mathrm{b} x]\right) = \dot{m}\,\mathrm{sgn}(P_\mathrm{c} - P_\mathrm{b})\,, \tag{2.76}$$

$$\frac{d}{dt}\left(\rho_\mathrm{b}[V_\mathrm{d} + A_\mathrm{b} x] c_\mathrm{v} T_\mathrm{b}\right) = -F_\mathrm{d}\dot{x} + \dot{m} c_\mathrm{p} T_\mathrm{u}\,\mathrm{sgn}(P_\mathrm{c} - P_\mathrm{b})\,, \tag{2.77}$$

$$m_\mathrm{d}\ddot{x} = A_\mathrm{b}\left(P_\mathrm{b} - P_\mathrm{atm} - P_\mathrm{f}(x, \dot{x})\right)\,, \tag{2.78}$$

$$P_i = \rho_i R T_i\,. \tag{2.79}$$

For the valve mass flow rate ($\dot{m}$) model, we assume that the barrel and chamber pressure are related by $\alpha(t)P_\mathrm{c} = P_\mathrm{b}$ where $0 < \alpha \leq 1$, so $P_\mathrm{c} \geq P_\mathrm{b}$ then the sgn functions can be eliminated and $T_\mathrm{u} = T_\mathrm{c}$. Also, $F_\mathrm{d} \equiv P_\mathrm{b} A_\mathrm{b}$.

Then, the mass conservation equations can be ignored, leaving us with

$$V_\mathrm{c}\frac{d}{dt}\left(\rho_\mathrm{c} c_\mathrm{v} T_\mathrm{c}\right) = -\dot{m} c_\mathrm{p} T_\mathrm{c}\,, \tag{2.80}$$

$$\alpha(t) P_\mathrm{c} = P_\mathrm{b}\,, \tag{2.81}$$

$$\frac{d}{dt}\left(\rho_\mathrm{b}[V_\mathrm{d} + A_\mathrm{b} x] c_\mathrm{v} T_\mathrm{b}\right) = -P_\mathrm{b} A_\mathrm{b}\dot{x} + \dot{m} c_\mathrm{p} T_\mathrm{c}\,, \tag{2.82}$$

$$m_\mathrm{d}\ddot{x} = A_\mathrm{b}\left[P_\mathrm{b} - P_\mathrm{atm} - P_\mathrm{f}\right]\,, \tag{2.83}$$

$$P_i = \rho_i R T_i\,. \tag{2.84}$$

Note that from the ideal gas law $\rho_i T_i = P_i / R$. Also note that $c_\mathrm{p} - c_\mathrm{v} = R$ and $c_\mathrm{p}/c_\mathrm{v} = \gamma$ so $c_\mathrm{v}/R = 1/(\gamma - 1)$.

Combined these state

$$c_\mathrm{v}\rho_i T_i = \frac{P_i}{\gamma - 1} \; . \tag{2.85}$$

I also define $V_\mathrm{b} \equiv V_\mathrm{d} + A_\mathrm{b} x$, which results in $\dot{V}_\mathrm{b} = A_\mathrm{b}\dot{x}$ and $\ddot{V}_\mathrm{b} = A_\mathrm{b}\ddot{x}$. Substituting all of these into the governing equations results in

$$V_\mathrm{c}\dot{P}_\mathrm{c} = -(\gamma-1)\dot{m}c_\mathrm{p}T_\mathrm{c} \; , \tag{2.86}$$

$$P_\mathrm{c} = \frac{1}{\alpha}P_\mathrm{b} \; , \tag{2.87}$$

$$\frac{d}{dt}(P_\mathrm{b}V_\mathrm{b}) = -(\gamma-1)P_\mathrm{b}\dot{V}_\mathrm{b} + (\gamma-1)\dot{m}c_\mathrm{p}T_\mathrm{c} \; , \tag{2.88}$$

$$\frac{m_\mathrm{d}}{A_\mathrm{b}^2}\ddot{V}_\mathrm{b} = P_\mathrm{b} - P_\mathrm{atm} - P_\mathrm{f} \; . \tag{2.89}$$

Substituting eqn. 2.86 into the last term of eqn. 2.88 and simplifying returns

$$\dot{P}_\mathrm{b}V_\mathrm{b} = -\gamma P_\mathrm{b}\dot{V}_\mathrm{b} - V_\mathrm{c}\dot{P}_\mathrm{c} \; . \tag{2.90}$$

Taking the time derivative of eqn. 2.87 returns

$$\dot{P}_\mathrm{c} = \frac{1}{\alpha}\dot{P}_\mathrm{b} + \frac{\dot{\alpha}}{\alpha^2}P_\mathrm{b} \; . \tag{2.91}$$

Substituting eqn. 2.91 into eqn. 2.90 and simplifying returns

$$\dot{P}_\mathrm{b}\left(V_\mathrm{b} + \frac{V_\mathrm{c}}{\alpha}\right) + P_\mathrm{b}\left(\gamma\dot{V}_\mathrm{b} + \frac{V_\mathrm{c}}{\alpha^2}\dot{\alpha}\right) = 0 \; . \tag{2.92}$$

Solving eqn. 2.89 for the barrel pressure and taking its time derivative leads to

$$P_\mathrm{b} = \frac{m_\mathrm{d}}{A_\mathrm{b}^2}\ddot{V}_\mathrm{b} + P_\mathrm{atm} + P_\mathrm{f} \; , \tag{2.93}$$

$$\dot{P}_\mathrm{b} = \frac{m_\mathrm{d}}{A_\mathrm{b}^2}\dddot{V}_\mathrm{b} + \dot{P}_\mathrm{f} \; . \tag{2.94}$$

These can be substituted into eqn. 2.92 to find a differential equation for V_b,

$$\left(\frac{m_\mathrm{d}}{A_\mathrm{b}^2}\dddot{V}_\mathrm{b} + \dot{P}_\mathrm{f}\right)\left(V_\mathrm{b} + \frac{V_\mathrm{c}}{\alpha}\right) + \left(\frac{m_\mathrm{d}}{A_\mathrm{b}^2}\ddot{V}_\mathrm{b} + P_\mathrm{atm} + P_\mathrm{f}\right)\left(\gamma\dot{V}_\mathrm{b} + \frac{V_\mathrm{c}}{\alpha^2}\dot{\alpha}\right) = 0 \; . \tag{2.95}$$

To get a reasonably simple exact solution to this equation, we assume $\dot{\alpha} = \dot{P}_\mathrm{f} = 0$ and non-dimensionalize with

$$V_\mathrm{b}^* \equiv \frac{V_\mathrm{b}}{A_\mathrm{b}L_\mathrm{b}} \quad , \quad \tau \equiv t\sqrt{\frac{A_\mathrm{b}P_\mathrm{atm}}{m_\mathrm{d}L_\mathrm{b}}} \quad , \quad \mathrm{CB} \equiv \frac{V_\mathrm{c}}{A_\mathrm{b}L_\mathrm{b}} \quad , \quad P_i^* \equiv \frac{P_i}{P_\mathrm{atm}} \quad , \quad V_\mathrm{d}^* \equiv \frac{V_\mathrm{d}}{A_\mathrm{b}L_\mathrm{b}} \; , \tag{2.96}$$

to find (after dropping the superscript * and using Newton's dot notation for $d/d\tau$)

$$\dddot{V}_\mathrm{b}\left(V_\mathrm{b} + \frac{\mathrm{CB}}{\alpha}\right) + \left(\ddot{V}_\mathrm{b} + 1 + P_\mathrm{f}\right)\gamma\dot{V}_\mathrm{b} = 0 \; . \tag{2.97}$$

This can be rewritten and integrated to get a third-order autonomous/time-invariant ordinary differential equation:

$$\frac{\dddot{V}_\mathrm{b}}{\ddot{V}_\mathrm{b}+1+P_\mathrm{f}} = -\gamma\frac{\dot{V}_\mathrm{b}}{V_\mathrm{b}+\frac{\mathrm{CB}}{\alpha}} \,, \tag{2.98}$$

$$A \equiv \ddot{V}_\mathrm{b}+1+P_\mathrm{f} \quad , \quad B \equiv V_\mathrm{b}+\frac{\mathrm{CB}}{\alpha} \,, \tag{2.99}$$

$$\frac{\dot{A}}{A} = -\gamma\frac{\dot{B}}{B} \longrightarrow \frac{d\log A}{d\tau}+\gamma\frac{d\log B}{d\tau} = 0 \longrightarrow \log A+\log B^\gamma = \log C \longrightarrow \left(\ddot{V}_\mathrm{b}+1+P_\mathrm{f}\right)\left(V_\mathrm{b}+\frac{\mathrm{CB}}{\alpha}\right)^\gamma = C \tag{2.100}$$

where C is a constant found from the initial conditions[2]:

$$V_\mathrm{b}(t=0) = V_\mathrm{d} \quad , \quad P_\mathrm{b}(t=0) = \ddot{V}_\mathrm{b}(t=0)+1+P_\mathrm{f}(t=0) = \alpha P_\mathrm{c}(t=0) = \alpha P_\mathrm{c,0} \,. \tag{2.101}$$

Thus,

$$C = \alpha P_\mathrm{c,0}\left(V_\mathrm{d}+\frac{\mathrm{CB}}{\alpha}\right)^\gamma \,, \tag{2.102}$$

and

$$\left(\ddot{V}_\mathrm{b}+1+P_\mathrm{f}\right)\left(V_\mathrm{b}+\frac{\mathrm{CB}}{\alpha}\right)^\gamma = \alpha P_\mathrm{c,0}\left(V_\mathrm{d}+\frac{\mathrm{CB}}{\alpha}\right)^\gamma \,. \tag{2.103}$$

This result is noticeably similar to eqn. 2.1, the starting point for the simple ideal barrel length equation. Better yet, this result can be used to find the ideal barrel length equation for this more general case. For the ideal case, the projectile stops accelerating, so $\ddot{V}_\mathrm{b} = 0$. At the end of the barrel, $V_\mathrm{b} = 1+V_\mathrm{d}$, so

$$(1+P_\mathrm{f})\left(1+V_\mathrm{d}+\frac{\mathrm{CB}}{\alpha}\right)^\gamma = \alpha P_\mathrm{c,0}\left(V_\mathrm{d}+\frac{\mathrm{CB}}{\alpha}\right)^\gamma \,, \tag{2.104}$$

which can be rewritten as

$$\mathrm{CB} = \frac{\alpha}{\left(\frac{\alpha P_\mathrm{c,0}}{1+P_\mathrm{f}}\right)^{1/\gamma}-1} - V_\mathrm{d}\alpha \,. \tag{2.105}$$

Eqn. 2.105 and eqn. 2.6 are remarkably similar, as they should be. The main difference is the effect of valve performance (α) to reduce the C:B ratio and that dead space (V_d) reduces the C:B ratio here instead of increasing it before.

If we define

$$C \equiv \alpha P_\mathrm{c,0}(V_\mathrm{d}+\mathrm{CB}\,/\alpha)^\gamma \quad , \quad D \equiv V_\mathrm{b}+\frac{\mathrm{CB}}{\alpha} \quad , \quad E \equiv 1+P_\mathrm{f} \tag{2.106}$$

then we can write eqn. 2.103 as

$$(\ddot{D}+E)D^\gamma = C \,, \tag{2.107}$$

and integrate using $\ddot{x}dx = \dot{x}d\dot{x}$ where x is any arbitrary variable:

$$\frac{1}{2}\dot{D}^2 = -\frac{CD^{1-\gamma}}{\gamma-1} - DE + F \,. \tag{2.108}$$

F is an arbitrary constant, which for the initial conditions previously used is

$$F = \left(V_\mathrm{d}+\frac{\mathrm{CB}}{\alpha}\right)\left(\frac{\alpha P_\mathrm{c,0}}{\gamma-1}+1+P_\mathrm{f}\right) \,, \tag{2.109}$$

2 All of this is non-dimensional.

and thus the dimensionless muzzle velocity (i.e., after setting $V_\mathrm{b}^* = 1 + V_\mathrm{d}^*$)

$$\frac{1}{2}\dot{V}_\mathrm{b}^2 = \frac{1}{2}\dot{x}^2 = \frac{\alpha P_{\mathrm{c},0}}{\gamma - 1}\left(V_\mathrm{d} + \frac{\mathrm{CB}}{\alpha}\right)\left[1 - \left(\frac{V_\mathrm{d} + \mathrm{CB}\,/\alpha}{1 + V_\mathrm{d} + \mathrm{CB}\,/\alpha}\right)^{\gamma-1}\right] - (1 + P_\mathrm{f})\,, \tag{2.110}$$

where I used the fact that $\dot{V}_\mathrm{b} = \dot{x}$ in non-dimensional form. Note that the effective pressure is $\alpha P_{\mathrm{c},0}$ and the effective C:B ratio is $\mathrm{CB}\,/\alpha$. This shows how valve performance changes overall performance.

In non-dimensional form, the energy efficiency (from eqn. 2.18) is

$$\eta = \frac{(\gamma - 1)\left(\frac{1}{2}\dot{x}^2\right)}{\mathrm{CB}\left[P_{\mathrm{c},0}^* - (P_{\mathrm{c},0}^*)^{1/\gamma}\right]}\,. \tag{2.111}$$

Plugging the equation for $\frac{1}{2}\dot{x}^2$ into the efficiency equation unfortunately does not seem to result in a consistent efficiency equation. The efficiency of the ideal C:B ratios with $\alpha = 1$ and $V_\mathrm{d} = P_\mathrm{f} = 0$ are not 1 as they should be (the process is reversible). Whether this is a mathematical mistake or a true feature of the model is not clear.

2.4.6 Effective initial chamber pressure

The α valve model requires an unphysical initial condition which does not necessarily require that gas mass and gas energy are equal to their values with less restrictive models. This is because the initial barrel pressure will not necessarily be atmospheric. If $P_\mathrm{atm} = \alpha P_{\mathrm{c},0}$ and/or $V_\mathrm{d} = 0$ then the α valve model is perfectly correct and mass and energy are consistent. Otherwise, an adjustment needs to be made to ensure that either the total mass or energy is consistent with its value without the α valve model. Energy consistency is preferred as η is the focus of this work, and without energy consistency η will not be bounded between 0 and 1.

Mass consistency. Let the variables with a hat over them refer to the effective state, i.e., $\widehat{P}_{\mathrm{c},0}$ is the effective pressure. Then by the ideal gas law

$$m_{\mathrm{c},0} = \frac{P_{\mathrm{c},0} V_\mathrm{c}}{R T_{\mathrm{c},0}} \quad , \quad m_{\mathrm{b},0} = \frac{P_{\mathrm{b},0} V_\mathrm{d}}{R T_\mathrm{atm}}\,, \tag{2.112}$$

and

$$\widehat{m}_{\mathrm{c},0} = \frac{\widehat{P}_{\mathrm{c},0} V_\mathrm{c}}{R T_{\mathrm{c},0}} \quad , \quad \widehat{m}_{\mathrm{b},0} = \frac{\alpha \widehat{P}_{\mathrm{c},0} V_\mathrm{d}}{R T_\mathrm{atm}}\,. \tag{2.113}$$

Setting $m_{\mathrm{c},0} + m_{\mathrm{b},0} = \widehat{m}_{\mathrm{c},0} + \widehat{m}_{\mathrm{b},0}$ returns

$$\widehat{P}_{\mathrm{c},0} = \frac{\dfrac{P_{\mathrm{c},0} V_\mathrm{c}}{T_{\mathrm{c},0}} + \dfrac{P_\mathrm{atm} V_\mathrm{d}}{T_\mathrm{atm}}}{\dfrac{V_\mathrm{c}}{T_{\mathrm{c},0}} + \dfrac{\alpha V_\mathrm{d}}{T_\mathrm{atm}}}\,. \tag{2.114}$$

Energy consistency. The original input energy,

$$E_0 = \frac{P_\mathrm{atm} V_\mathrm{c}}{\gamma - 1}\left[P_{\mathrm{c},0}^* - (P_{\mathrm{c},0}^*)^{1/\gamma}\right]\,, \tag{2.115}$$

is just chamber input energy as the barrel starts out at atmospheric pressure. Therefore, if we break up the energy into $\widehat{E}_{\text{c}}$ and $\widehat{E}_{\text{b}}$ where

$$\widehat{E}_{\text{c}} = \frac{P_{\text{atm}} V_{\text{c}}}{\gamma - 1} \left[\widehat{P}^{*}_{\text{c},0} - (\widehat{P}^{*}_{\text{c},0})^{1/\gamma} \right] , \tag{2.116}$$

$$\widehat{E}_{\text{b}} = \frac{P_{\text{atm}} V_{\text{d}}}{\gamma - 1} \left[\alpha \widehat{P}^{*}_{\text{c},0} - (\alpha \widehat{P}^{*}_{\text{c},0})^{1/\gamma} \right] , \tag{2.117}$$

we can solve the implicit equation $E_0 = \widehat{E}_{\text{c}} + \widehat{E}_{\text{b}}$ to find $\widehat{P}^{*}_{\text{c},0}$.

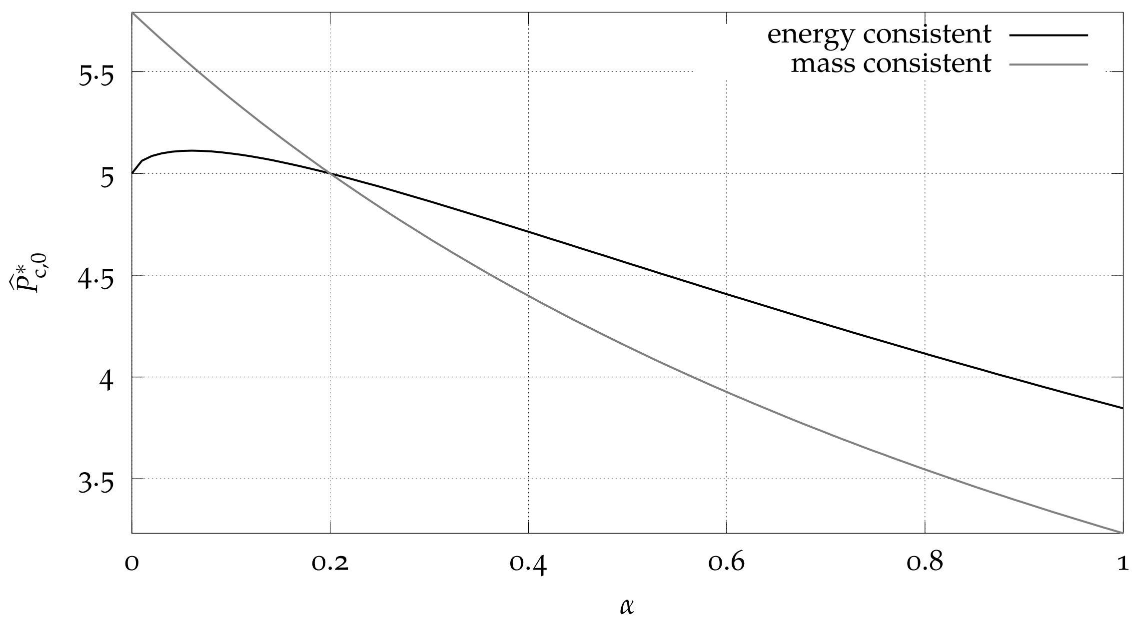

Figure 3: Effective pressures for the mass and energy consistent cases for $P^{*}_{\text{c},0} = 5$ with $V^{*}_{\text{d}} = 0.1$ and $\text{CB} = 0.2$.

Compensating for this effect does not help the inconsistency with the $\alpha = 1$ case as the model is wrong even with $V_{\text{d}} = 0$ there too (no compensation is needed for $V_{\text{d}} = 0$).

Notes about and shortcomings of this analytical model.

- This model predicts that friction universally reduces performance, which is not observed in numerical simulations. Barrel friction can improve performance slightly. The hypothesized mechanism is static friction keeping the projectile stationary, allowing the barrel pressure to increase higher because the projectile has not moved. If the projectile moves prematurely, it increases the barrel volume and thus decreases the pressure, reducing acceleration. Dart mass and dead volume are hypothesized to have similar effects as well.
- The valve model prevents proper modeling of the initial state of the gas in the barrel. Consequently, it is possible for this model to predict energy efficiencies over 100%. Adjusting $P_{\text{c},0}$ to account for this can solve this problem.
- This model does not differentiate between static and dynamic friction. The friction force is constant and directed in the negative x direction. Consequently, the projectile will slowly accelerate backwards at the initial time.
- The strong dependence of dart mass on energy efficiency is not seen in this model. The most likely explanation for this is that the valve coefficient also contains information about the dart mass.

- The valve model is not entirely realistic. The pressure ratio P_b/P_c starts out high and will move towards 1 in most cases.
- The energy efficiency η strongly depends on α — a 5% change in α results in a change in η of over 30%. Worse, α depends strongly on the initial chamber pressure. $\alpha P^*_{\text{c},0} \approx$ constant for $P^*_{\text{c},0}$ less than about 6 based on available test data [Tre11b]. This constant of proportionality might be a more useful "model constant".

Deriving α from valve flow equations. The ISO valve model for subsonic flow (eqn. 2.68) can be used to find the precise meaning of α outside of its definition, $\alpha \equiv P_\text{b}/P_\text{c}$. The laminar and sonic models are not captured well by the α model, but at least the α model gets the dynamics qualitatively correct.

2.4.7 Computer simulation of pneumatic guns

Numerical simulation approaches can be slightly more complex than the previously detailed analytical approaches by numerically integrating ODEs. I've developed several simulations for pneumatics and springers. One popular simulation for Windows is GGDT [Hal08], which unfortunately uses an outdated and inaccurate valve model, but is nonetheless accurate enough for general use.

They also can be far more complex and rigorous, taking a computational fluid dynamics (CFD) approach. Of the different CFD approaches, I plan to try a few:

- the Lagrangian approach developed by LANL much like what Jacobs [Jac98] did, or
- an Eulerian approach with moving boundaries.

2.5 Spring gun ballistics

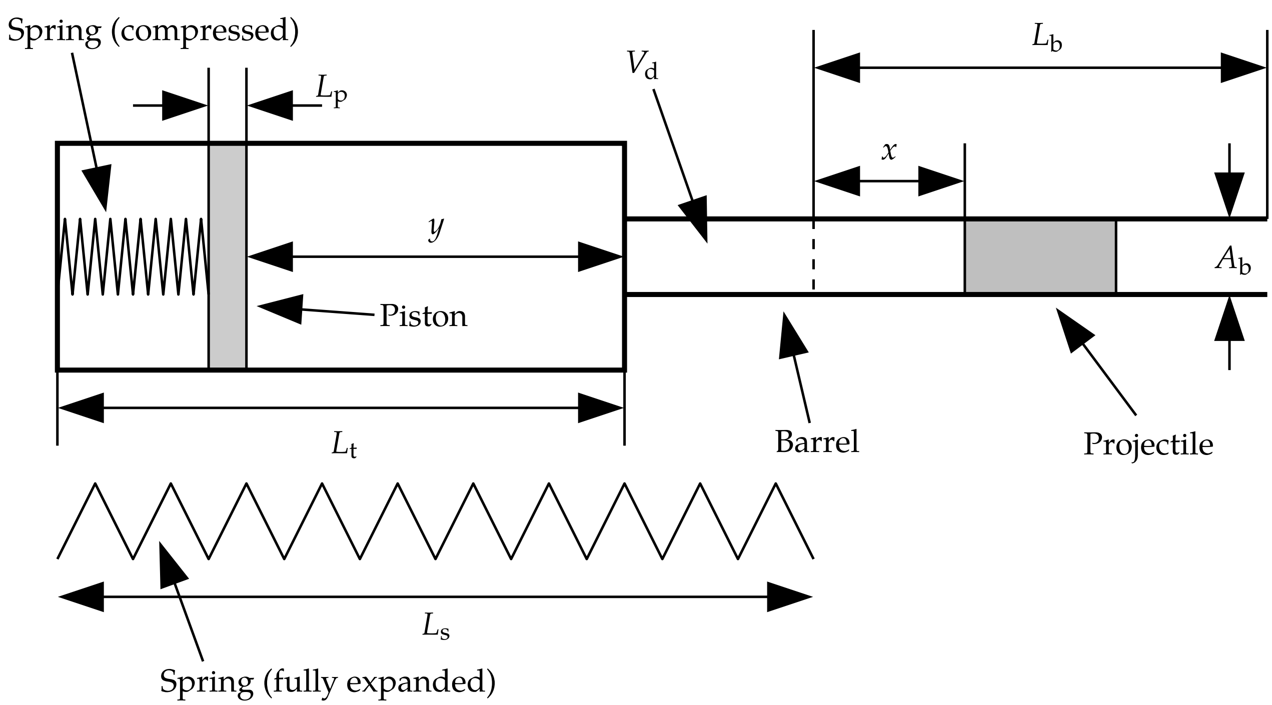

Figure 4: Some geometric variables and components of the model spring gun.

2.5.1 Equations governing spring guns

2.5.2 Analytical lumped-parameter model for springers

$$m_{\mathrm{p}}\ddot{y} = -k(L_{\mathrm{s}} - L_{\mathrm{t}} + L_{\mathrm{p}} + y) + A_{\mathrm{p}}(P_{\mathrm{p}} - P_{\mathrm{atm}} - P_{\mathrm{fp}}) \tag{2.118}$$

$$\frac{d}{dt}(A_{\mathrm{p}}y\rho_{\mathrm{p}}T_{\mathrm{p}}c_{\mathrm{v}}) = -P_{\mathrm{p}}A_{\mathrm{p}}\dot{y} - \dot{m}c_{\mathrm{p}}T_{\mathrm{u}} \tag{2.119}$$

$$\alpha P_{\mathrm{p}} = P_{\mathrm{b}} \tag{2.120}$$

$$\frac{d}{dt}\left[\rho_{\mathrm{b}}(V_{\mathrm{d}} + A_{\mathrm{b}}x)c_{\mathrm{v}}T_{\mathrm{b}}\right] = -P_{\mathrm{b}}A_{\mathrm{b}}\dot{x} + \dot{m}c_{\mathrm{p}}T_{\mathrm{u}} \tag{2.121}$$

$$m_{\mathrm{d}}\ddot{x} = A_{\mathrm{b}}(P_{\mathrm{b}} - P_{\mathrm{atm}} - P_{\mathrm{fb}}) \tag{2.122}$$

$$\rho_i RT_i = P_i \tag{2.123}$$

$$V_{\mathrm{p}} \equiv A_{\mathrm{p}}y \quad , \quad V_{\mathrm{b}} \equiv V_{\mathrm{d}} + A_{\mathrm{b}}x \quad , \quad c_{\mathrm{v}}\rho_i T_i = \frac{P_i}{\gamma - 1} \tag{2.124}$$

$$\cancel{\dot{V}_{\mathrm{p}}P_{\mathrm{p}} + V_{\mathrm{p}}\dot{P}_{\mathrm{p}}} + V_{\mathrm{p}}\dot{P}_{\mathrm{p}} = -(\gamma - \cancel{1})P_{\mathrm{p}}\dot{V}_{\mathrm{p}} - (\gamma - 1)\dot{m}c_{\mathrm{p}}T_{\mathrm{u}} \tag{2.125}$$

$$(\gamma - 1)\dot{m}c_{\mathrm{p}}T_{\mathrm{u}} = -\left[\gamma P_{\mathrm{p}}\dot{V}_{\mathrm{p}} + \dot{P}_{\mathrm{p}}V_{\mathrm{p}}\right] \tag{2.126}$$

$$\cancel{P_{\mathrm{b}}\dot{V}_{\mathrm{b}}} + \dot{P}_{\mathrm{b}}V_{\mathrm{b}} = -(\gamma - \cancel{1})P_{\mathrm{b}}\dot{V}_{\mathrm{b}} + (\gamma - 1)\dot{m}c_{\mathrm{p}}T_{\mathrm{u}} \tag{2.127}$$

$$\gamma P_{\mathrm{b}}\dot{V}_{\mathrm{b}} + \dot{P}_{\mathrm{b}}V_{\mathrm{b}} = \underbrace{(\gamma - 1)\dot{m}c_{\mathrm{p}}T_{\mathrm{u}}}_{\text{substitute in eqn. 2.126}} \tag{2.128}$$

$$\gamma(P_{\mathrm{b}}\dot{V}_{\mathrm{b}} + P_{\mathrm{p}}\dot{V}_{\mathrm{p}}) + (\dot{P}_{\mathrm{b}}V_{\mathrm{b}} + \dot{P}_{\mathrm{p}}V_{\mathrm{p}}) = 0 \tag{2.129}$$

$$\tag{2.130}$$

Then, substitute in eqn. 2.120,

$$\gamma P_{\mathrm{b}}\left(\dot{V}_{\mathrm{b}} + \frac{1}{\alpha}\dot{V}_{\mathrm{p}}\right)\dot{P}_{\mathrm{b}}\left(V_{\mathrm{b}} + \frac{1}{\alpha}V_{\mathrm{p}}\right) = 0\,. \tag{2.131}$$

$$P_{\mathrm{p}} = \frac{m_{\mathrm{p}}}{A_{\mathrm{p}}^2}\ddot{V}_{\mathrm{p}} + \frac{k}{A_{\mathrm{p}}}(L_{\mathrm{s}} - L_{\mathrm{t}} + L_{\mathrm{p}} + y) + P_{\mathrm{p}} - P_{\mathrm{atm}} - P_{\mathrm{fp}} \tag{2.132}$$

$$P_{\mathrm{b}} = \frac{m_{\mathrm{d}}}{A_{\mathrm{b}}^2}\ddot{V}_{\mathrm{b}} + P_{\mathrm{atm}} + P_{\mathrm{fd}} \tag{2.133}$$

$$\mathrm{CB} \equiv \frac{A_{\mathrm{p}}L_{\mathrm{o}}}{A_{\mathrm{b}}L_{\mathrm{b}}} \quad , \quad \tau \equiv t\sqrt{\frac{A_{\mathrm{b}}P_{\mathrm{atm}}}{m_{\mathrm{d}}L_{\mathrm{b}}}} \quad , \quad P_i^* \equiv \frac{P_i}{P_{\mathrm{atm}}} \quad , \quad m_{\mathrm{p}}^* = \frac{m_{\mathrm{p}}}{m_{\mathrm{d}}}\frac{L_0}{L_{\mathrm{b}}}\frac{A_{\mathrm{b}}}{A_{\mathrm{p}}} \quad , \quad k^* \equiv \frac{kL_0}{P_{\mathrm{atm}}A_{\mathrm{p}}} \quad , \quad L_{\mathrm{st}}^* \equiv \frac{L_{\mathrm{s}} - L_{\mathrm{t}} + L_{\mathrm{p}}}{L_0} \tag{2.134}$$

Dropping the * superscript and using the dot notation for $d/d\tau$ returns

$$0 = \gamma P_\mathrm{b} \left(\dot{V}_\mathrm{b} + \frac{\mathrm{CB}}{\alpha} \dot{V}_\mathrm{p} \right) + \dot{P}_\mathrm{b} \left(V_\mathrm{b} + \frac{\mathrm{CB}}{\alpha} V_\mathrm{p} \right) \,, \tag{2.135}$$

$$P_\mathrm{p} = m_\mathrm{p} \ddot{V}_\mathrm{p} + k(L_\mathrm{st} + V_\mathrm{p}) + 1 + P_\mathrm{fp} \,, \tag{2.136}$$

$$P_\mathrm{b} = \ddot{V}_\mathrm{b} + 1 + P_\mathrm{fb} \,. \tag{2.137}$$

Integrating eqn. 2.135 leads to

$$\left(V_\mathrm{b} + \frac{\mathrm{CB}}{\alpha} V_\mathrm{p} \right)^\gamma P_\mathrm{b} = C \tag{2.138}$$

where C is a constant which can be found from the initial conditions

$$V_\mathrm{b}(t=0) = V_\mathrm{d} \quad , \quad V_\mathrm{p}(t=0) = 1 \quad , \quad P_\mathrm{b}(t=0) = \alpha \tag{2.139}$$

to be

$$C = \alpha \left(V_\mathrm{d} + \frac{\mathrm{CB}}{\alpha} \right)^\gamma \,. \tag{2.140}$$

And thus

$$(\ddot{V}_\mathrm{b} + 1 + P_\mathrm{fb}) \left(\frac{V_\mathrm{b} + \frac{\mathrm{CB}}{\alpha} V_\mathrm{p}}{V_\mathrm{d} + \frac{\mathrm{CB}}{\alpha}} \right)^\gamma = \alpha \,. \tag{2.141}$$

This equation can be solved to find an ideal C:B ratio equation,

$$\mathrm{CB} = \alpha \left(\frac{1 - V_\mathrm{d} \left[\left(\frac{\alpha}{1 + P_\mathrm{fd}} \right)^{1/\gamma} - 1 \right]}{\left(\frac{\alpha}{1 + P_\mathrm{fd}} \right)^{1/\gamma} - V_\mathrm{pf}} \right) \,, \tag{2.142}$$

where V_pf is the volume of the plunger tube when the dart leaves the barrel. This equation is similar to the ideal C:B ratio equation found earlier from adiabatic process relationships. Unfortunately, V_pf is not known a-priori. Usually people assume this volume is zero as a first approximation. For the absolutely ideal case, we expect plunger volume to equal zero when the dart leaves, but for a particular ideal C:B ratio we have no reason to suspect that is true.

This assumption does not seem to hold based on available high speed footage as seen in figure 5. The plunger volume when the dart leaves is not always negligible.

A single ODE for the barrel volume, like in the pneumatic case, can be derived. Use eqn. 2.136, eqn. 2.137, and eqn. 2.120 to find that

$$\alpha \left[m_\mathrm{p} \ddot{V}_\mathrm{p} + k(L_\mathrm{st} + V_\mathrm{p}) + 1 + P_\mathrm{fp} \right] = \ddot{V}_\mathrm{b} + 1 + P_\mathrm{fb} \,. \tag{2.143}$$

Then solve eqn. 2.141 for V_P,

$$V_\mathrm{p} = \frac{\alpha}{\mathrm{CB}} \left[\left(\frac{\alpha}{\ddot{V}_\mathrm{b} + 1 + P_\mathrm{fb}} \right)^{1/\gamma} (V_\mathrm{d} + \mathrm{CB}/\alpha) - V_\mathrm{b} \right] \,, \tag{2.144}$$

differentiate it twice and plug the result into eqn. 2.143 to find (after simplification) something that is too complicated for me to even bother putting here.

This approach does not seem to be tractable.

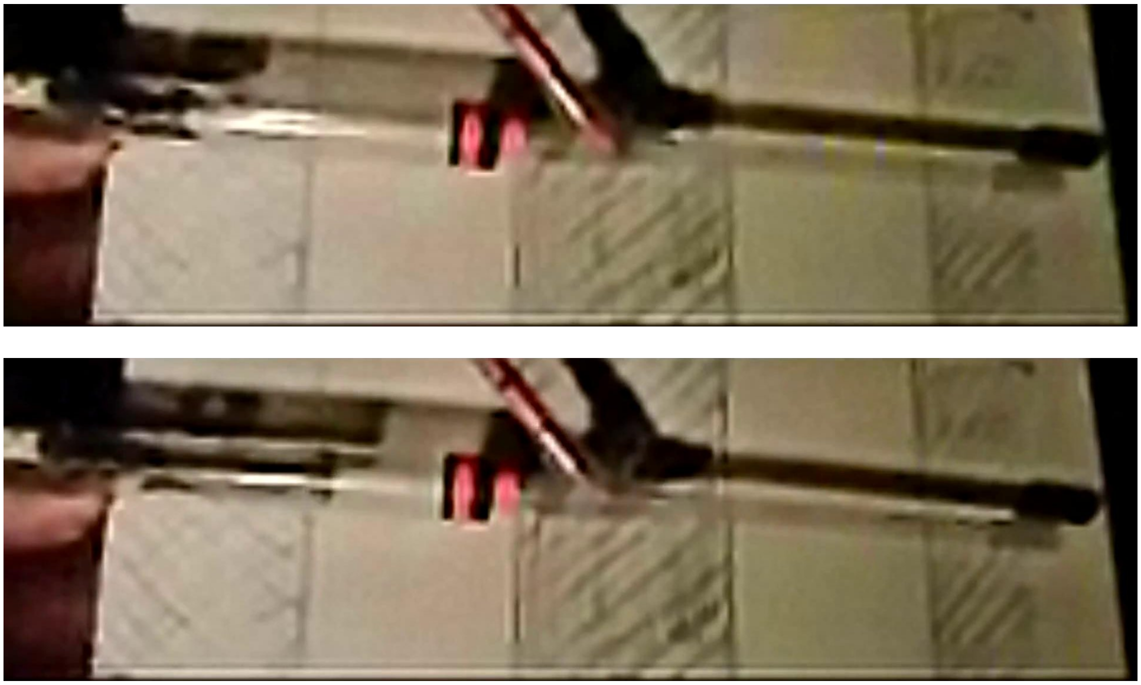

Figure 5: The position of the plunger and (less clearly) the dart in a springer can be seen in this high-speed footage of a clear springer by AToMATRoN [AT011]. The top is before firing; the bottom is approximately when the dart leaves the barrel.

Estimates for the final plunger volume. Two estimates are worth noting. The first, that the plunger volume is zero when the dart leaves, is probably true for the most ideal of the ideal cases. The second uses eqn. 2.136, eqn. 2.137, and eqn. 2.120 to find the plunger volume when the dart stops accelerating. If we assume that the plunger stops accelerating and is moving at a constant (or zero) velocity when the dart leaves that barrel, we can find that

$$V_{\mathrm{pf}} = \frac{(1 + P_{\mathrm{fd}})/\alpha - (1 + P_{\mathrm{fp}})}{k} - L_{\mathrm{st}} \,. \tag{2.145}$$

This assumption appears to be approximately satisfied in figure 5 and is probably reasonable for barrels that are not too short.

2.5.3 Computer simulation of springers

2.6 Dead space

Popular conception is that dead space only decreases performance. This is a good rule of thumb, but it's not strictly true. A small amount of dead space helps performance if the flow from the plunger or gas chamber to the barrel is sufficiently restricted, though the amount is negligibly small.

2.7 Dart-barrel fit and friction

Static friction is usually helpful to performance, and *dynamic* friction usually hurts performance.

2.8 Energy budget

2.9 Pump and plunger friction

2.10 Spring hysteresis

2.11 Leaks

2.11.1 Around the dart

2.11.2 Valve pilot

3 External ballistics

In this section I develop a simple analytical model for the trajectory of a Nerf dart and use this model to find range, "energy efficiency", and time to reach a target. I use the common "flat-fire" assumption, which allows for the solution of the ODEs.

The basic equations of external ballistics have remained unsolved in a useful closed form since then were formulated over 300 years ago. This is not really a major issue in applied math, as you could *define* the solution to these equations as some function much like how sin and cos are essentially defined as the solutions to some simple vibratory differential equations.

Despite the recent media hype, a teenager did not obtain a useful solution [Mat12]. He obtained a series solution. Similar series solutions were obtained far earlier, and series solutions are not particularly useful as even numerical approximations are easier to use. Here I obtain an algebraic solution to a simplified approximate set of external ballistic equations that is appropriate to use for Nerf darts.

For more information about basic external ballistic models, I suggest reading the appropriate chapters of McCoy [McC99] and Carlucci and Jacobson [CJ07]. My use of the Lambert W function also follows from the similar external ballistic model of Warburton, Wang, and Burgdörfer [WWB10].

3.1 Dart trajectory (small angle approximation)

3.1.1 Equations of motion

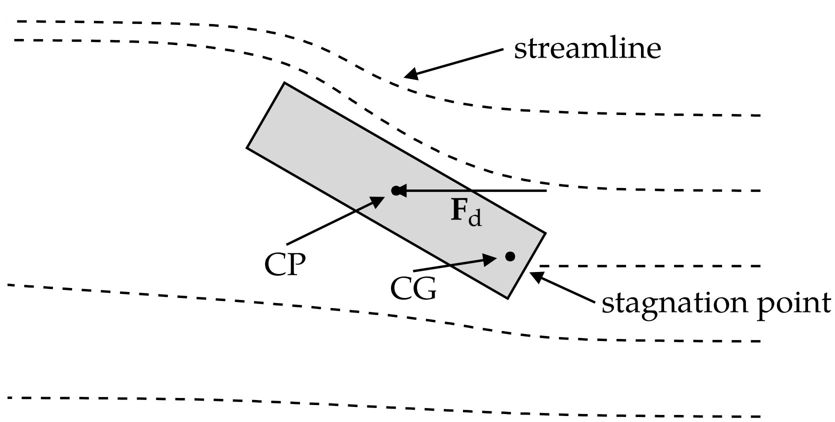

Figure 6: Front-weighted dart in motion with CP (center of pressure), CG (center of gravity), $\mathbf{F}_\mathrm{d}$ (drag force vector), approximate streamlines, and the stagnation point labeled.

For simplicity, I treat the dart as a point mass. This assumption is acceptable for stable darts, that is, darts that do not continuously overturn. Darts that are sufficiently weighted in the front are expected to be stable, as I will explain.

Darts rotate about their center of gravity. In figure 6 you can see a representation of a dart in motion with approximate streamlines, which represent the path air particles take. This is a front-weighted dart that has begun to overturn by rotating around its center of gravity, which is in the front. The air that runs along the top of the dart applies a drag force in opposition to the velocity of the dart (which is purely to the right in this figure). As darts rotate about their center of gravity, the back end of the dart will rotate counter-clockwise until it is on the other side, which is when the process reverses. Damping losses ensure that this process does not continue indefinitely (it moves smaller angles in each cycle), and eventually the dart will be traveling straight again.

In short, front-weighted darts have a natural tendency to align themselves with their velocity vector. This tendency allows the analysis of the system to be simplified considerably as the yaw (the angle the dart makes with its velocity vector) can be assumed to be zero and thus the dart can be treated as a point mass.

The quadratic law of drag is assumed and written below. The law assumes that the drag force is a function of the speed of the dart squared. Testing so far indicates that this is a good assumption for Nerf darts as their drag coefficients are fairly constant under this law.

$$\mathbf{F}_\mathrm{d} = -\tfrac{1}{2}\rho_\mathrm{atm} A C_\mathrm{d} \mathbf{v} \left|\left|\mathbf{v}\right|\right| \tag{3.1}$$

$\mathbf{F}_\mathrm{d}$ is the drag force vector. ρ_atm is atmospheric air density, which is approximately 1.28 kg/m^3. C_d is the drag coefficient. A is the cross-sectional area of the dart, which in our case is simply $\frac{1}{4}\pi D^2$ where D is the diameter of the dart. $\mathbf{v}$ is the velocity vector of the dart.

Newton's second law (below) will be used to determine the accelerations of the dart.

$$\sum \mathbf{F} = m\mathbf{a} = \mathbf{F}_\mathrm{d} - mg\mathbf{j} \tag{3.2}$$

m is the dart mass. g is acceleration due to Earth's gravity, which is approximately 9.81 m/s^2.

For clarify, the frame of reference will be described. The problem is restricted to a 2D Cartesian plane. The x-direction is in the direction the darts are fired. The y-direction is up. The point $(0, 0)$ approximately coincides with the feet of the shooter. The point $(0, h)$ is precisely where the dart is fired from. h is the height of the dart when it is fired. R is the range of the dart.

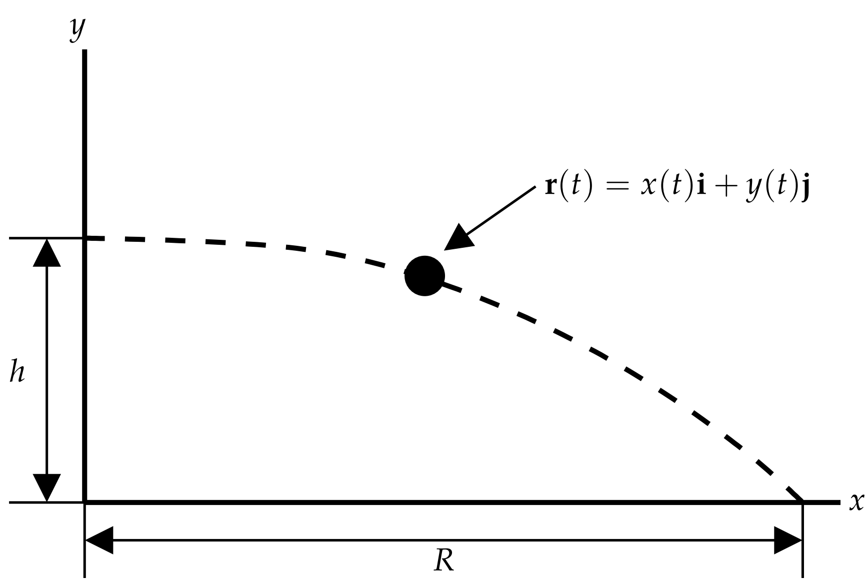

Figure 7: The coordinate system and trajectory of the dart.

$\mathbf{r}(t)$ is the trajectory of the dart; is and its time derivatives (velocity $\mathbf{v}(t)$ and acceleration $\mathbf{a}(t)$) are written below.

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} \longrightarrow \mathbf{v} = \dot{x}\mathbf{i} + \dot{y}\mathbf{j} \longrightarrow \mathbf{a} = \ddot{x}\mathbf{i} + \ddot{y}\mathbf{j} \tag{3.3}$$

The speed (length of the velocity vector) is shown below for those unfamiliar with the vector norm notation. This is also known as the speed.

$$||\mathbf{v}|| = \sqrt{\dot{x}^2 + \dot{y}^2} \tag{3.4}$$

All of the above are plugged together to get a vector equation of motion which is decomposed into the x and y components below.

$$m(\ddot{x}\mathbf{i} + \ddot{y}\mathbf{j}) = -\tfrac{1}{2}\rho_{\text{atm}} A C_{\text{d}}(\dot{x}\mathbf{i} + \dot{y}\mathbf{j})\sqrt{\dot{x}^2 + \dot{y}^2} - mg\mathbf{j} \tag{3.5}$$

$$m\ddot{x} = -\tfrac{1}{2}\rho_{\text{atm}} A C_{\text{d}}\dot{x}\sqrt{\dot{x}^2 + \dot{y}^2} \tag{3.6}$$

$$m\ddot{y} = -\tfrac{1}{2}\rho_{\text{atm}} A C_{\text{d}}\dot{y}\sqrt{\dot{x}^2 + \dot{y}^2} - mg \tag{3.7}$$

To simplify the math, a constant K is defined.

$$K \equiv \frac{\rho_{\text{atm}} A C_{\text{d}}}{2m} \tag{3.8}$$

$$\ddot{x} = -K\dot{x}\sqrt{\dot{x}^2 + \dot{y}^2} \tag{3.9}$$

$$\ddot{y} = -K\dot{y}\sqrt{\dot{x}^2 + \dot{y}^2} - g \tag{3.10}$$

The equations above are the governing differential equations of the motion of a point mass with quadratic drag.

3.1.2 Initial conditions

At time $t = 0$, the dart has a velocity $\mathbf{v}(0) = V_0(\cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j})$ where θ is the firing angle above flat. V_0 is the muzzle velocity of the gun. This equation means that $\dot{x}(0) = V_0\cos\theta$ and $\dot{y}(0) = V_0\sin\theta$. Also, the location of the dart at time $t = 0$ is $\mathbf{r}(t) = h\mathbf{j}$. This means that $x(0) = 0$ and $y(0) = h$.

3.1.3 "Flat fire" approximation

The simplification. The problem with equations of motion (equations 3.9 and 3.10) is that they are a system of non-linear coupled ordinary differential equations. Some solutions are known, but they generally are given in terms of integrals rather than so-called "elementary functions." As they are integrals with no known analytical solution, they are generally worth no more than a numerical solution. But I do not want a numerical solution, so I must make a simplification to the system if I intend to solve it.

The speed of the dart, $||\mathbf{v}||$, is dominated by $\dot{x}$. If fired flat, the dart moves far faster (and farther in distance) in the x-direction. Is approximating $||\mathbf{v}|| = \sqrt{\dot{x}^2 + \dot{y}^2}$ by $\dot{x}$ a good approximation? It is, of course, or else I would not mention it, but how good of an approximation is it?

Upper bound of $\dot{y}$. To determine how good of an approximation my change is, I must find an upper bound for $\dot{y}$. The case without drag is always faster than the case with drag. So I will find the maximum y-velocity without drag.

I start with the acceleration in the y-direction without drag. I integrate this and apply the same initial conditions I mentioned earlier.

$$\ddot{y}_{\text{nd}} = -g \longrightarrow \dot{y}_{\text{nd}} = -gt + C_1 \longrightarrow \dot{y}_{\text{nd}}(0) = 0 = C_1 \tag{3.11}$$

$$y_{\text{nd}}(t) = -\frac{gt^2}{2} + C_2 \longrightarrow y_{\text{nd}}(0) = h = C_2 \longrightarrow y_{\text{nd}}(t) = h - \frac{gt^2}{2} \tag{3.12}$$

$$\tag{3.13}$$

The maximum y-velocity occurs right before the dart hits the ground. So I will find the time of the impact. The subscript "i,nd" means impact, no drag here.

$$y_{\text{nd}}(t_{\text{i,nd}}) = 0 \longrightarrow \frac{gt_{\text{i,nd}}^2}{2} = h \longrightarrow t_{\text{i,nd}} = \sqrt{\frac{2h}{g}} \tag{3.14}$$

With the impact time now known, I can plug the time into the velocity equation to get the maximum velocity and square it to find the upper bound of $\dot{y}^2$.

$$\dot{y}(t_{\text{i,nd}}) = -g\sqrt{\frac{2h}{g}} = -\sqrt{2gh} \longrightarrow \dot{y}^2 \leq 2gh \tag{3.15}$$

A tall man would hold his gun at approximately a height of 2 m off the ground. This means that $\dot{y} \leq 6.3\,\text{m/s}$.

Error committed through approximation. What are the limits of this approximation's validity? I will define the error and calculate what value of $\dot{x}$ attains a 5% and 25% error.

$$e \equiv \left|1 - \frac{\dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2}}\right| = 1 - \frac{\dot{x}}{\sqrt{\dot{x}^2 + 39.226}} \tag{3.16}$$

This leads to a 5% error in the speed term when $\dot{x} = 19.05\,\text{m/s}$ (62.5 ft/s) and a 25% error when $\dot{x} = 7.10\,\text{m/s}$ (23.3 ft/s). These velocities are quite low, and they will not be seen even at the ends of most trajectories unless the effects of drag are very significant. And even then, as this term would only have a significant error for part of the shot, this approximation may still be acceptable overall.

Consequently, this approximation is perfectly acceptable as long as we make sure that the x-velocity remains within the acceptable bounds. If there are any questions about whether or not this approximation is acceptable, a numerical solution of the full ODEs (equations 3.9 and 3.10) can also be checked.

Simplified equations of motion.

$$\ddot{x} = -K\dot{x}^2 \tag{3.17}$$

$$\ddot{y} = -K\dot{x}\dot{y} - g \tag{3.18}$$

The equations of motion are now uncoupled. Eqn. 3.17 is still a non-linear ODE, but eqn. 3.18 is a linear ODE after the solution for $\dot{x}$ is substituted in. The exact solution for these equations can be found.

3.1.4 Solution for $\dot{x}(t)$ and $x(t)$

After some searching, I found a change of variables that reduced eqn. 3.17 to a first-order ODE.

$$z \equiv \dot{x}^2 \longrightarrow \dot{z} = 2\dot{x}\ddot{x} \longrightarrow \ddot{x} = \frac{\dot{z}}{2\dot{x}} = \frac{\dot{z}}{2\sqrt{z}} \tag{3.19}$$

I then substituted in the old variables in terms of the new variables to get a new ODE that I can solve. I then proceeded to solve this ODE.

$$\frac{\dot{z}}{2\sqrt{z}} = -Kz \longrightarrow \dot{z} = -2Kz^{3/2} \longrightarrow z^{-3/2} = -Kdt \longrightarrow \frac{z^{-1/2}}{-\frac{1}{2}} = -2kt - 2C_1 \tag{3.20}$$

$$z^{-1/2} = K + C_1 \longrightarrow z^{-1/2} = (\dot{x}^2)^{-1/2} = \frac{1}{\dot{x}} = Kt + C_1 \longrightarrow \dot{x} = \frac{1}{Kt + C_1} \tag{3.21}$$

Now I apply the IC $\dot{x}(0) = V_0 \cos\theta$, rearrange the solution for $\dot{x}$ and integrate it and apply the IC $x(0) = 0$ to get the solution for x.

$$\dot{x}(t) = \frac{V_0 \cos\theta}{V_0 \cos\theta\, Kt + 1} \tag{3.22}$$

$$x(t) = \frac{\log(V_0 \cos\theta\, Kt + 1)}{K} + C_2 \tag{3.23}$$

$$x(0) = \cancelto{0}{\frac{\log(1)}{K}} + C_2 = 0 \tag{3.24}$$

$$x(t) = \frac{\log(V_0 \cos\theta\, Kt + 1)}{K} \tag{3.25}$$

Proof of equivalence with dragless case. Some readers might wonder how eqn. 3.25 for x can possibly be equivalent to the no-drag case as K goes to zero. $\dot{x}$ and $\ddot{x}$ more obviously reduce to the no-drag equations as $K \to 0$. The check for x relies on the Taylor series for the natural log.

$$\log(1+z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} z^n \quad , \quad |z| \leq 1 \tag{3.26}$$

Using $z = V_0 \cos\theta\, Kt$ allows us to check this directly.

$$\lim_{C_d \to 0} x(t) = \lim_{K\to 0} x(t) = \lim_{K \to 0} \frac{1}{K} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (V_0 \cos\theta\, Kt)^n \quad , \quad |V_0 \cos\theta\, Kt| \leq 1 \tag{3.27}$$

$$= \lim_{K \to 0} \frac{V_0 \cos\theta \cancel{K} t}{\cancel{K}} + \cancelto{0}{\frac{1}{K} \sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n} (V_0 \cos\theta\, Kt)^n} = V_0 \cos\theta\, t \quad \checkmark \tag{3.28}$$

The sum from 2 to ∞ is zero because every term contains K. The result is precisely what is expected without drag—a constant velocity. This is because there is no way for the dart to slow down. Thus, the equations for x and $\dot{x}$ are perfectly consistent with the no-drag case.

3.1.5 Solution for $\dot{y}(t)$ and $y(t)$

Eqn. 3.18 can be rewritten as follows with the expression for $\dot{x}$ substituted in. I define $A \equiv V_0 \cos\theta\, K$ to simplify the equation slightly.

$$\ddot{y} = \frac{-A\dot{y}}{At+1} - g \tag{3.29}$$

I use the change of variables $\xi \equiv \log(At+1)$ and change the ODE above to one that can be directly integrated[3]. The subscript here represents differentiation with respect to the variable in the subscript. Two subscripts is double differentiation.

$$\xi = \log(At+1) \longrightarrow \xi_t = \frac{A}{At+1} \longrightarrow \xi_{tt} = \frac{-A^2}{(At+1)^2} \tag{3.30}$$

$$y_t = y_\xi \xi_t \longrightarrow y_{tt} = (y_\xi \xi_t)_t = y_{\xi\xi}(\xi_t)^2 + y_\xi \xi_{tt} \tag{3.31}$$

The ODE can now be rewritten in terms of the new replacement for t, ξ. Note that $e^\xi = At+1$.

$$y_{\xi\xi}(\xi_t)^2 + y_\xi \xi_{tt} = -\frac{A}{At+1} y_\xi \xi_y - g \tag{3.32}$$

$$y_{\xi\xi}\left(\frac{A}{At+1}\right)^2 - \cancel{y_\xi\left(\frac{A}{At+1}\right)^2} = \cancel{-y_\xi\left(\frac{A}{At+1}\right)^2} - g \tag{3.33}$$

$$y_{\xi\xi} = -g\left(\frac{At+1}{A}\right)^2 = -\frac{g}{A^2}e^{2\xi} \tag{3.34}$$

This equation can be directly integrated. It solution and the conversion back to t is below.

$$y(\xi) = -\frac{g}{4A^2}e^{2\xi} + B\xi + C \tag{3.35}$$

$$y(t) = -\frac{g(At+1)^2}{4A^2} + B\log(At+1) + C \tag{3.36}$$

Now I apply the ICs $y(0) = h$ and $\dot{y}(0) = V_0 \sin\theta$.

$$y(0) = -\frac{g}{4A^2} + C = h \longrightarrow C = h + \frac{g}{4A^2} \tag{3.37}$$

$$\dot{y} = \frac{-g(At+1)}{2A} + \frac{BA}{At+1} \tag{3.38}$$

$$\dot{y}(0) = V_0 \sin\theta = -\frac{g}{2A} + BA \longrightarrow B = \frac{g}{2A^2} + \frac{\tan\theta}{K} \tag{3.39}$$

This leads to the following solution.

3 Despite the fact that as $A \to 0$ here ξ will no longer change as t changes, the change of variables still arrives at the correct solution for no drag.

$$y(t) = h + \frac{g}{4A^2} + \left[\frac{g}{2A^2} + \frac{\tan\theta}{K}\right] \log(At+1) - \frac{g(At+1)^2}{4A^2} \tag{3.40}$$

This simplifies to the following equation.

$$y(t) = h + \left[\frac{g}{2(KV_0\cos\theta)^2} + \frac{\tan\theta}{K}\right] \log(KV_0\cos\theta\, t+1) - \frac{gt^2}{4} - \frac{gt}{2KV_0\cos\theta} \tag{3.41}$$

3.1.6 Equation for range

Implicit analytical solution for R. Note that eqn. 3.41 can be rewritten with eqn. 3.25 as the following.

$$y(t) = h + \left[\frac{g}{2(KV_0\cos\theta)^2} + \frac{\tan\theta}{K}\right] Kx(t) - \frac{gt^2}{4} - \frac{gt}{2KV_0\cos\theta} \tag{3.42}$$

This will allow me solve for t_i, the time of impact with the ground. If I had not eliminated the logarithm, this time would have to be found numerically; no explicit algebraic solution in elementary functions would exist. Once this time is found, it can be substituted into eqn. 3.25 to find the range.

By definition, $x(t_\mathrm{i}) = R$ and $y(t_\mathrm{i}) = 0$. When these are substituted into the equation above, I get the following.

$$0 = h + \left[\frac{g}{2(KV_0\cos\theta)^2} + \frac{\tan\theta}{K}\right] KR - \frac{gt_i^2}{4} - \frac{gt_i}{2KV_0\cos\theta} \tag{3.43}$$

This can be solved for t_i with the quadratic formula. The correct solution is below. The other solution is negative and consequently non-physical.

$$t_i = \frac{\sqrt{2KR + \dfrac{4(KV_0\cos\theta)^2(h+R\tan\theta)}{g} + 1} - 1}{KV_0\cos\theta} \tag{3.44}$$

This equation can be rearranged into a form that is easily substituted into eqn. 3.25.

$$V_0K\cos\theta\, t_i + 1 = \sqrt{\frac{4(KV_0\cos\theta)^2(h+R\tan\theta)}{g} + 2KR + 1} \tag{3.45}$$

$$x(t_i) = R = \frac{\log(V_0K\cos\theta\, t+1)}{K} \tag{3.46}$$

This leads to the following *implicit* equation for R.

$$2KR = \log\left(\frac{4(KV_0\cos\theta)^2(h+R\tan\theta)}{g} + 2KR + 1\right) \tag{3.47}$$

A new parameter called β is now defined along with the Froude number, Fr.

$$\beta \equiv \frac{4h(KV_0)^2}{g} \equiv \frac{h}{g}\left(\frac{\rho_\mathrm{atm}C_\mathrm{d}AV_0}{m}\right)^2 \quad , \quad \mathrm{Fr} \equiv \frac{V_0^2}{gh} \tag{3.48}$$

As will be explained, β is very important for the flat-fire case. β tells you how much drag affects your darts, but here it also controls the range. The Froude number controls the relative effect of gravity on the dart.

3.1.7 Range energy efficiency η_r

Motivation and definition. How much of the kinetic energy put into the dart is actually used to get the range it does? A certain amount is wasted by drag—the remainder is what is used. Energy efficiency is essentially "what you get" divided by "what you pay". Darts "get" the result of the no drag case but pay more kinetic energy. Consequently, I define what I call "range energy efficiency": $\eta_r \equiv T_{nd}/T$.

T_{nd} is the kinetic energy that would have been required to achieve a range R without drag (this makes the muzzle velocity V_0 lower—m is unchanged) *fired flat*. This, again, is basically what you get out of shooting a dart. T, on the other hand, is what you pay—the actual kinetic energy of the dart. The fired flat part is important, as it is necessary for simplicity (fired at the optimal angle can not be found analytically).

What is the muzzle velocity required to get a range R without drag? What is the kinetic energy required? And what is the kinetic energy required with drag?

$$R = x_{nd}(t_{i,nd}) = V_{0,nd} t_{i,nd} = V_{0,nd}\sqrt{\frac{2h}{g}} \longrightarrow V_{0,nd}^2 = \frac{gR^2}{2h} \tag{3.49}$$

$$T_{nd} = \frac{1}{2} m V_{0,nd}^2 = \frac{mgR^2}{4h} \tag{3.50}$$

$$T = \frac{1}{2} m V_0^2 \tag{3.51}$$

Now I substitute the above into the definition of η_r to find a more useful form of it:

$$\eta_r = \frac{\frac{1}{4}(mgR^2)/h}{\frac{1}{2} m V_0^2} \equiv \frac{gR^2}{2hV_0^2} . \tag{3.52}$$

This is the new definition for η_r in the flat-fire case.

Implicit analytical solution for η_r. The new definition of η_r can be solved for R and substituted into eqn. 3.47 to find an implicit equation for η_r.

$$R = V_0 \sqrt{\frac{2h\eta_r}{g}} \tag{3.53}$$

$$2KV_0\sqrt{\frac{2h\eta_r}{g}} = \log\left(\frac{4h(KV_0)^2}{g} + 2KV_0\sqrt{\frac{2h\eta_r}{g}} + 1\right) \tag{3.54}$$

This leads to a relatively simple implicit equation for η_r as a function of β. Note that η_r is a function of only three parameters (β, Fr, and θ), and only one parameter (β) when $\theta = 0$. A functional form this simple is not what I would have expected from dimensional analysis alone.

$$\sqrt{2\beta\eta_r} = \log\left[\beta(1 + \sqrt{2\,\mathrm{Fr}\,\eta_r}\tan\theta) + \sqrt{2\beta\eta_r} + 1\right] \tag{3.55}$$

The range efficiency η_r is plotted for a wide variety of βs typical for Nerf below for $\theta = 0$.

Explicit exact solutions. The Lambert W function can be used to write an explicit solution for η_r assuming that $\theta = 0$. Warburton, Wang, and Burgdörfer [WWB10] derived a similar expression for small angles

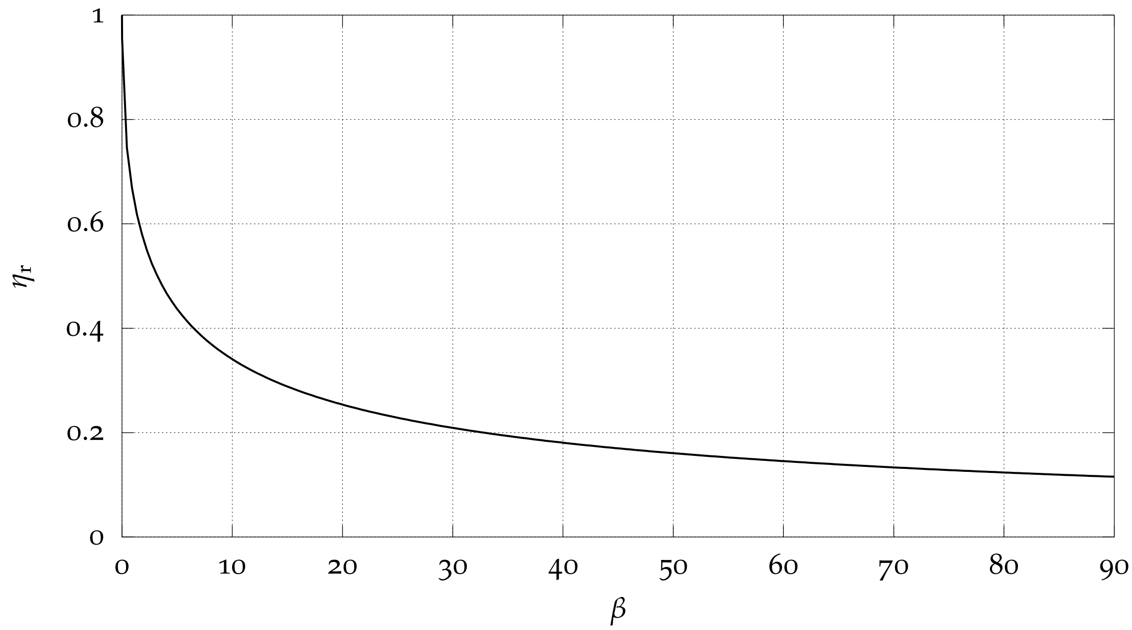

Figure 8: Range efficiency, η_r, as a function of β for $\theta = 0$. Note that because $\theta = 0$, this plot is bounded between 0 and 1 in η_r.

without a starting height. Including both angles and a starting height does not seem to be possible to get an explicit exact solution. The lower branch of the W function is the physical branch.

The Lambert W function is defined as the W that satisfies

$$z = W(z) \exp W(z) \; . \tag{3.56}$$

The task here is to put eqn. 3.55 into this form to find W.

Raise e to the power of eqn. 3.55,

$$e^{\sqrt{2\beta\eta_r}} = \beta + 1 + \sqrt{2\beta\eta_r} \; . \tag{3.57}$$

Now define

$$A \equiv \beta + 1 \quad , \quad B \equiv \sqrt{2\beta\eta_r} \; , \tag{3.58}$$

and substitute into eqn. 3.57 to find

$$e^B = A + B \longrightarrow e^A e^B = e^{A+B} = e^A(A+B) \longrightarrow -e^{-A} = -(A+B)e^{-(A+B)} \; , \tag{3.59}$$

which implies

$$W[-e^{-A}] = -(A+B) \longrightarrow B = -W[-e^{-A}] - A \; . \tag{3.60}$$

Rewriting the result above returns

$$\eta_r = \frac{(W[-\exp(-\beta-1)] + \beta + 1)^2}{2\beta} \; . \tag{3.61}$$

Conclusions for design. η_r tells you how much drag affects your dart. Note how quickly η_r drops as β increases. This shows that an enormous amount of energy is lost due to drag and this translates into a significant reduction in range. Thankfully, it's not just drag coefficient alone that matters—I can go through each variable and explain how we can increase range or range efficiency (to decrease required energy to prime a blaster).

We have little to no control over the firing height (h), gravitational acceleration (g), or atmospheric air density (ρ_{atm}). Consequently, we shouldn't focus on changing these variables.

This relationship does show us that lower drag darts (as measured by $C_d A$) do have longer ranges. This, of course, is no surprise. We should possibly investigate smaller diameter darts and definitely investigate lower drag coefficient darts.

What is surprising, however, is that the ratio of muzzle velocity to dart mass (V_0/m) can be reduced to improve range. No one tries to manipulate this parameter in Nerf guns, and these results indicate that people should reduce this parameter to improve range. Keep in mind that very slow darts are unlikely to be useful in a Nerf war. Perhaps the take-home message should be that if you keep kinetic energy constant and reduce V_0/m, range will probably increase.

Range ratio ψ_r. Some readers might have thought of my choice for "range efficiency" was unusual. "The actual ranges are what matters" is what someone might claim. They might propose a new measure of how efficient a firing configuration is that I'll call the range ratio: $\psi_r \equiv R/R_{\text{nd}}$. R is the range with muzzle velocity V_0 and drag. R_{nd} is the range with muzzle velocity V_0 and no drag. R_{nd} is always greater than or equal to R because darts without drag travel further. So $\psi_r \leq 1$ and ψ_r tells us how much of the total possible range our dart is reaching.

How is this measure of efficiency different? It's not—up to a difference in the power, at least. Below I'll calculate the range without drag and plug this into the definition for ψ_r to show that $\psi_r^2 \equiv \eta_r$.

$$R_{\text{nd}} = x_{\text{nd}}(t_{\text{i,nd}}) = V_0 t_{\text{i,nd}} = V_0 \sqrt{\frac{2h}{g}} \tag{3.62}$$

$$\psi_r \equiv \frac{R}{R_{\text{nd}}} = \sqrt{\frac{g}{2h}} \frac{R}{V_0} = \sqrt{\frac{gR^2}{2hV_0^2}} = \sqrt{\eta_r} \tag{3.63}$$

With the idea of the range ratio in mind, the actual range can be calculated for any system where η_r (or equivalently, ψ_r) is known by the definition of ψ_r and its relationship to η_r.

$$R = V_0 \sqrt{\frac{2h\eta_r}{g}} = V_0 \psi_r \sqrt{\frac{2h}{g}} \tag{3.64}$$

Whether to prefer η_r or ψ_r is a matter of personal preference.

The explicit solution for ψ_r is

$$\psi_r = -\frac{W[-\exp(-\beta - 1)] + \beta + 1}{\sqrt{2\beta}} \tag{3.65}$$

3.1.8 Estimating muzzle velocity from range

3.2 Drag coefficient of a Nerf dart

Beaver [Bea12] did some external ballistic tests with homemade Nerf darts (dart length was 1.25 in). Analysis of his data indicated that the drag coefficient of a Nerf dart is on average 0.67 ± 0.029, consistent with the data compiled Hoerner [Hoe65, p. 3–12], shown in figure 9.

Earlier tests I did [Tre11a] indicated a far lower drag coefficient, but these tests likely were not actually fired flat. Beaver [Bea12] used a level to ensure his barrel was flat during firing.

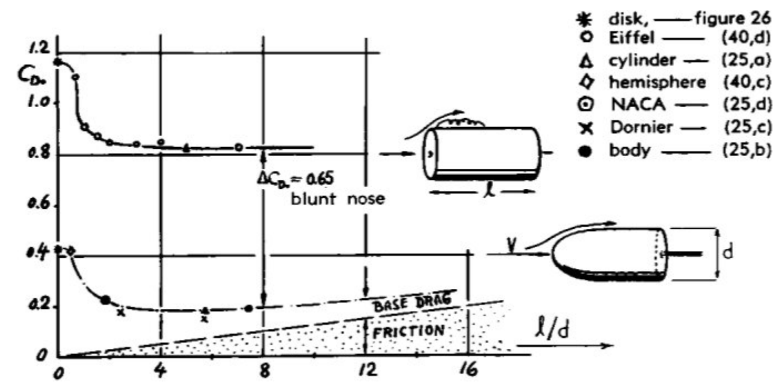

Figure 9: The drag coefficient of blunt nose and streamlined cylinders as a function of length divided by diameter as compiled by Hoerner [Hoe65, p. 3–12].

3.3 Time to target

The time to target is how long a dart takes to reach its target distance. This result thusfar has not been considered at all in Nerf gun design, yet it is probably one of the most important.

The time to target, t_t can be found from eqn. 3.25:

$$x(t) = \frac{\log(V_0 K t + 1)}{K} \,. \tag{3.25}$$

If we define R_t as the target range we can find that

$$t_\mathrm{t} = \frac{\exp(K R_\mathrm{t}) - 1}{V_0 K} \,. \tag{3.66}$$

Eqn. 3.66 is the time it takes a Nerf dart to reach a target a distance of R_t away if it is fired straight at it with minimal affects from gravity.

KED limits. If you have a set KED limit, you can derive an equation for the time to target as a function of dart mass:

$$t_\mathrm{t} = \frac{8}{\rho_\mathrm{atm} d^3 C_\mathrm{d}} \sqrt{\frac{2}{E_k''}} \left(\frac{m_\mathrm{d}}{\pi}\right)^{3/2} \left[\exp\left(\frac{\pi \rho_\mathrm{atm}}{8} \frac{d^2 C_\mathrm{d} R_\mathrm{t}}{m_\mathrm{d}}\right) - 1\right] \,. \tag{3.67}$$

The idea here is to choose the dart mass that allows the time to target to be shorter than someone's response time, which the sum of reaction time and movement time. If the time to target is shorter than this time, and if you aim well, you guarantee a hit. The average reaction time is about 190 ms [**wikipedia_contributors_mental_????**]. You can estimate the movement time by estimating an average distance someone needs to move to miss your dart (about 20 cm seems reasonable) and the average speed they move at (4.5 m/s might be a good upper bound).

Figure 10 shows the time to target of a typical Nerf dart as a function of dart mass if the KED limit is 10 mJ/mm^2 and you are aiming for a target 9 m away.

Rate of fire and your firing accuracy.

3.4 Optimal firing angle

The "optimal" firing angle maximizes range.

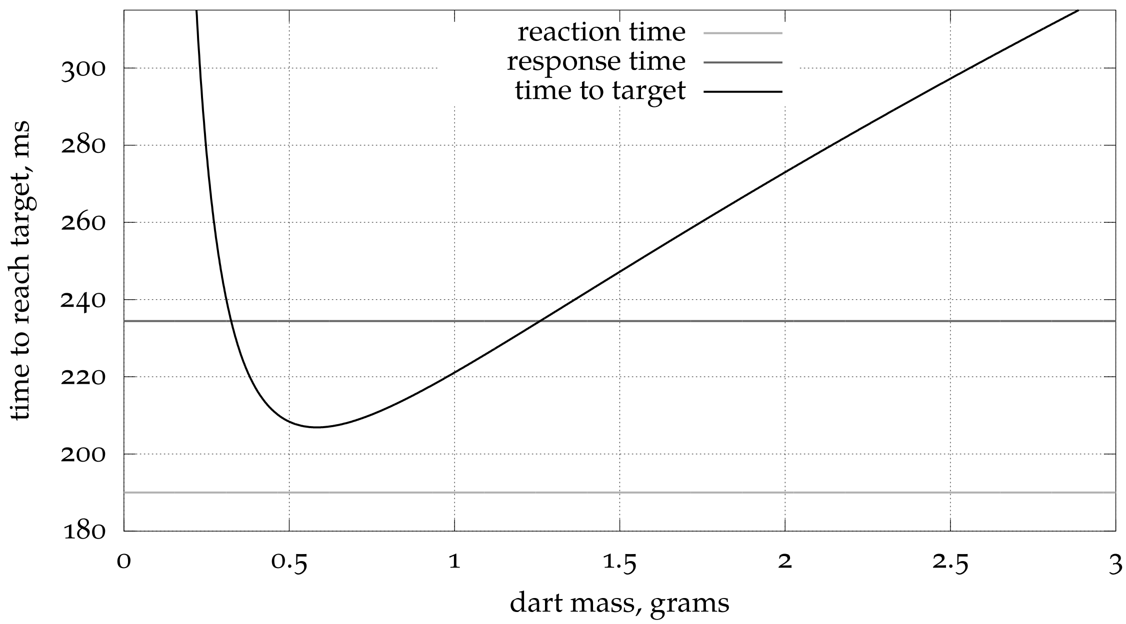

Figure 10: Time to target as a function of dart mass for a particular case.

3.4.1 Case without drag

An implicit algebraic equation for the optimal firing angle (i.e., the angle which maximizes range) without drag can be found. While not directly applicable to most ballistics problems, this result is useful for understanding what affects the optimal firing angle. The initial conditions and governing equations are

$$\ddot{x} = 0\,, \tag{3.68}$$
$$\ddot{y} = -g\,, \tag{3.69}$$
$$x(0) = 0\,, \tag{3.70}$$
$$y(0) = h\,, \tag{3.71}$$
$$\dot{x}(0) = V_0 \cos\theta\,, \tag{3.72}$$
$$\dot{y}(0) = V_0 \sin\theta\,, \tag{3.73}$$

where h is the starting height of the projectile, V_0 is the muzzle velocity, and θ is the firing angle above flat. The solution of these equations is

$$x = V_0 \cos\theta\, t\,, \tag{3.74}$$
$$y = V_0 \sin\theta\, t - \tfrac{1}{2} g t^2 + h\,. \tag{3.75}$$

The range is defined as where the projectile hits the ground, hence where $y = 0$. Thus,

$$0 = -\tfrac{1}{2} g t_{\text{impact}}^2 + V_0 \sin\theta\, t_{\text{impact}} + h\,, \tag{3.76}$$

which can be solved for t_{impact} to find

$$t_{\text{impact}} = \frac{V_0 \sin\theta + \sqrt{V_0^2 \sin^2\theta + 2gh}}{g}\,. \tag{3.77}$$

The range is found by substituting this time into the equation for x which leads to

$$R^* \equiv \frac{R\,g}{V_0^2} = \cos\theta \left(\sin\theta + \sqrt{\sin^2\theta + 2/\mathrm{Fr}} \right) , \tag{3.78}$$

where Fr is the Froude number, $\mathrm{Fr} \equiv V_0^2/(gh)$.

The optimal firing angle is found by differentiating the equation above with respect to θ, leading to

$$0 = \cos^2\theta \left(\frac{\sin\theta}{\sqrt{\sin^2\theta + 2/\mathrm{Fr}}} + 1 \right) - \sin\theta \left(\sqrt{\sin^2\theta + 2/\mathrm{Fr}} + \sin\theta \right) . \tag{3.79}$$

This equation can be solved numerically to find the optimal firing angle. This angle is plotted in figure 11 as a function of the Froude number. As shown, the angle approaches 45° as Fr increases.

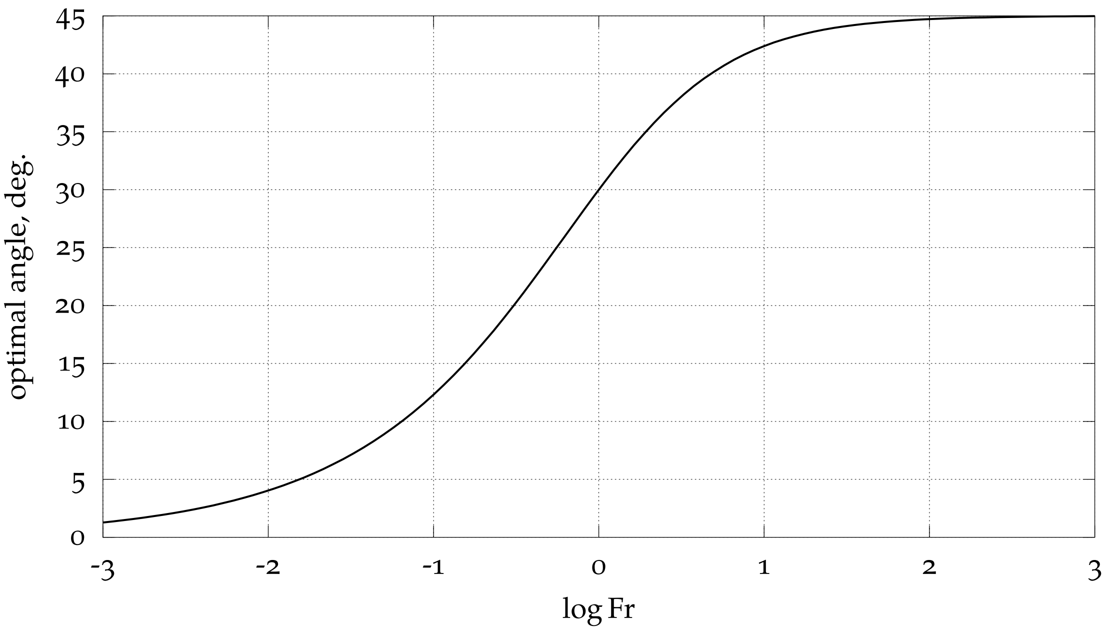

Figure 11: Optimal firing angle as a function of Froude number for the dragless case.

3.4.2 Case with drag

3.5 Accuracy and precision

Difference between accuracy and precision. Accuracy and precision are related concepts that are often confused. *Accuracy* is generally how close a measurement is to the true value, or in our case how close a dart lands to the intended target. *Precision* is the spread of the measurement, or in this case, the spread of multiple firings of darts.

3.5.1 Accuracy and precision

What is most central to accuracy and precision is repeatability. We want the trajectories to be as consistent as possible between shots. Consequently, we'll be looking to eliminate sources of variation.

Dart stability seems to be necessary for high accuracy and high precision. Stability here refers to resistance against overturning or perhaps even fishtailing. If a dart overturns of fishtails significantly, it is subject to aerodynamic forces that are inconsistent from shot to shot, so the accuracy and precision are affected greatly. The range is generally also very significantly affected.

We know that front-weighted darts have their center of gravity far in front of the center of pressure (where the net aerodynamic forces act), and this leads to stable darts. The darts rotate about their center of gravity, so when the center of gravity is basically at the tip of the dart, the dart basically can only move the length of the dart into the air stream. This applies a force that will rotate the dart back until it is parallel to the flow. This is why front-weighted darts are stable. If they started to become unstable, they'd correct themselves.

There are many other sources of variation that are worth looking in to. We want to ensure that darts are consistent. A few things to keep consistent: dart masses, weight distributions, and drag characteristics. The weight distribution can be kept consistent by using the same materials for all darts and carefully positioning and sizing all of the dart materials. The drag characteristics basically are shape of the dart; we don't want some darts to be bent, have different tips, have hot glue sticking off one end for some darts, etc. Here's the short version: making better and more consistent darts will help accuracy and precision.

Another source of variation is muzzle velocity. If your gun shoots strong on one shot and weak on the next, then obviously where you aimed for the first shot is probably not where your dart will land for the second. Muzzle velocity can be made more consistent by identifying sources of variation in the gun-dart system. Dart mass must be kept consistent here again. If a valve is used, it's better to use a piloted valve as they open basically instantaneously and the time to open is relatively independent of how quickly the user pulls the trigger. If a spring gun is being engineered, try to design the system to make the catch release at the same rate regardless of how quickly the user pulls the trigger. I could go on; you get the idea.

The actual dart mass is also important in addition to its consistency. Heavier darts are more resistant to wind and other transient forces that will push your darts off course. This, unfortunately, conflicts with some people's safety guidelines, but perhaps those guidelines can change in the future.

The actual muzzle velocity is also important for similar reasons. Higher muzzle velocities can lead to lower transit time (the time from leaving the barrel until hitting the target) for the dart, which should reduce the total impulse applied to the dart from transient forces like wind. And as we know, lower impulses mean lower changes in momentum, which means the dart will be more on target. Of course, the muzzle velocities could increase so much that drag causes the transit time to increase. I think this is an area where we need to do some testing before figuring out what works well. Increasing muzzle velocity could potentially counteract some of the limitations in accuracy and precision seen with dart mass restrictions.

Aerodynamic darts might be more accuracy or precise. More aerodynamic darts can maintain their speed better than other darts, so their transit time is reduced, which should reduce the total perturbative impulse (as previously mentioned) from wind, etc., that will push them off course.

The Spud Wiki's page on spudgun accuracy [Spu08b] notes that barrel vibrations can cause inaccuracies. I recall this being mentioned on some real gun ballistics websites too. I'm not too sure how significant this effect is for Nerf, but it might be more pronounced for more flexible barrel materials like PETG. A few tests to see if the barrel stiffness (the product of the modulus of elasticity and the second moment of area) has an effect on accuracy seem to be justified. There is a table listing the stiffnesses of some common barrel materials in the materials section of this document.

A number of other things on the gun can be changed to improve accuracy and precision. Sights are an obvious addition. Simple iron sights are plenty adequate for most Nerf guns, though, I am intrigued by the possibility of using ladder sights with angled shots. The sight radius, the distance between the front and back sights [Sho09], can also be increased to allow you to be more precise when aiming. Longer sight radii make aiming errors more noticeable.

Extra air shot out of the barrel when the dart leaves the barrel is called muzzle blast. If the air is slightly unbalanced (and it will be due to turbulence), this could have a negative effect on accuracy and precision because the tail of the dart will be bumped. This could cause what is called "fishtailing". Porting is one way to avoid this problem [Wik12]. Another way is to use barrels that are just long enough for the dart to leave without excess gas escaping. Thankfully, this coincides with the barrel length that maximizes performance (for low-speed guns, at least), so using the optimal barrel length can improve accuracy and precision.

Barrel length brings to mind another item in the myth category along with rifling for Nerf. Some people seem to think that longer barrels somehow make things more accurate or precise. This is probably true to some degree, that is, barrels that are shorter than the dart probably aren't very accurate. Some people seem to think that constraining the darts to move in a straight line helps somehow. But this is based on a poor understanding of Newtonian mechanics. When the dart leaves, it's a ballistic projectile. Where it was in the past is irrelevant; the conditions when it leaves are all that matters.

Once the barrel is long enough, then its length does not influence accuracy or precision unless increases in length change the sight radius, vibration characteristics (i.e., natural frequency) of the barrel, muzzle blast, etc. And it's obvious that longer is not always better here.

3.5.2 Summary of ways to improve accuracy and precision

- Front-weight darts to make them stable.
- Make darts consistently. Keep the shape consistent. Keep the mass consistent (and measure it to check).
- Make darts well.
- Use more aerodynamic darts.
- Use systems that remove human variation from the gun, like piloted valve systems (i.e. "backpressure tanks").
- Increase the dart mass.
- Increase the muzzle velocity.
- Use a stiff barrel.
- Use a sight.
- Use the barrel length that maximizes performance.

Safety, efficiency, and other concerns may prevent satisfying all of these ways to improve accuracy and precision.

3.6 Center of gravity

3.7 Center of pressure

3.8 Side wind

3.9 Dart stability

3.10 Muzzle blast

3.10.1 Why rifling does not help Nerf darts

Some Nerfers have proposed that rifled barrels may be beneficial for Nerf blasters by making an analogy with real guns. But is this true? I will examine the two most popular claimed benefits of rifling, that rifling increases range and improves accuracy, and conclude that rifling as implemented thus far has had no significant effect on range or accuracy and it is not likely to have any effect for weighted darts under any circumstances.

First, the reader must realize that these claims are made most often without any backing. The hypothesis that rifling improves accuracy or range is often made based on misunderstandings of what rifling does. Spinning projectiles do not have less drag. Projectiles are spun to improve stability, as I will explain.

Stability of projectiles. A projectile is stable if it flights straight without overturning. This is desirable as the overturning motion reduces accuracy and range.

Rifling is used to improve the stability of a projectile's flight. But can the stability of a Nerf dart be improved? In general, the answer is no because Nerf darts get their stability from static rather than dynamic characteristics of the dart.

The simplest way to make a stable projectile is to put the center of gravity far in front of the center of pressure. Details as to why this is stable will be added later. Most Nerf darts get their stability in this way; this is why darts are weighted at their nose.

But, most real bullets are made of a single material and they do not have this desirable weight distribution. Spinning the bullet around its longitudinal axis (as rifling does) can stabilize bullets in this case.

So, by simple examination of the mechanisms involved, we can conclude that rifling won't have any significant effect on darts with the right weight distribution. Those darts are already very aerodynamically stable. There is no reason to rifle as there will not be any real benefit.

Some benefit from rifling seems plausible for very light darts that do not have the right weight distribution. But this is not an argument for rifling necessarily; adding weight to the front is by far the easiest way to stabilize these projectiles. However, this may not seem to be an acceptable choice for some Nerfers. Very lightweight darts may be desirable for safety reasons, however, there are other ways to improve safety of a dart (like reducing the muzzle velocity) that are far simpler than rifling.

Potential disadvantages of rifling There are many potentially significant disadvantages to rifling that most proponents of the idea are unaware of. I detail the disadvantages that come to mind below.

- Increased friction — If done poorly, the rifling could increase friction in the barrel and potentially reduce performance as a consequence.
- Leaks around projectile — If done poorly, the rifling grooves could allow for air to leak around the projectile, reducing performance.
- Increased complexity of building — Smoothbore barrels are simpler.
- Less translational KE — To have a spinning dart, some of the energy that would have been put into translational kinetic energy and have contributed to range is instead put into rotational kinetic energy. Rifling is beneficial when this trade-off improves stability such that range or accuracy is improved satisfactorily. However, the reduction in translational KE may not be acceptable in all cases.
- Reduction of stability — Poorly made darts may not have their weight distributed evenly around the longitudinal axis of the dart. Spinning could destabilize these darts and reduce range and accuracy.

Examining the accuracy claim with data. In 2009, a Nerfer who went by the handle Landru did some tests to see what effect spinning a dart had on accuracy [Lan09]. He used a setup with a spinning barrel. He believed that this spinning barrel provides a way to control the spinning without making multiple rifled barrels. The test did not address rifling directly, rather, it addressed the question of whether spinning darts could even improve accuracy.

Landru posted some data that he claimed showed that the standard deviation of the locations of darts spun at 2000 RPM was lower than that from no spinning.

However, Landru neglected any sort of statistical analysis. I made a brief post that demonstrated his methods were flawed. I used an f-test to see whether there was any statistically significant difference between the two groups. Assuming a sample size of 20, I found critical f-values of 0.46 and 2.12 for $\alpha = 10\%$. The f-value of was 1.49. As this was between the critical values, the differences were not statistically significant and consequently we can not determine if they were due to the rifling or random chance.

Landru made no follow-up tests.

Examining the range claim with data. Back in perhaps 2003, a Nerfer who went by the handle Vassili tested rifled PETG barrels [Vas03]. He found that the average range of rifled PETG was higher than that of smoothbore PETG. Thankfully, Vassili didn't claim rifling improved range directly. He only offered a tautology: "When it worked, it worked." But did it work? Can we attribute any of the differences to the rifling and not random chance?

No, we can not. A t-test suggests the two data sets are statistically the same at the $\alpha = 5\%$ level. The critical t-value is 2.65. The t-value of the test for the mean is 1.21. As this is within the bounds we would expect at the 5% level of error, we can confidently state that rifling did not increase range in this case.

However, it can be shown that rifling increases the standard deviation of the range with an f-test (data to be added later). This should lead to a decrease in precision due to a decrease in repeatability (each shot is more variable). It also shows that more shots will have lower range with rifling. These two disadvantages are significant.

Further, it is unknown whether these shots were angled and if so, whether the angle was kept constant. If Vassili was simplying eye-balling the angle, this is another reason to doubt the results.

Conclusion. Based on the implausibility of the explanation for the benefit for rifling and the lack of evidence to suggest that rifling provides any benefit for Nerf darts, I conclude that rifling is ineffective at best and harmful at worst for Nerf.

Particularly light Nerf darts could potentially benefit from spinning, however, this has not been shown to date, and there is no convincing reason not to front weight them instead.

3.11 Alternative ammo

4 Design

4.1 Turrets

4.2 Sights

4.3 Ergonomics

4.4 Pumps

4.5 Darts

5 Materials

5.1 Nerf gun material failure case studies

5.2 Fracture mechanics

5.3 Fatigue

5.4 Buckling

5.5 Adhesives

5.6 Impact design

5.7 Stress concentrators

5.8 Barrels

In this section I summarize many important material properties and characteristics of barrel materials. For the moment I focus solely on the barrels with diameters of about 0.53 in because these are the most popular.

Material	part #	price $/ft	OD (in)	ID (in)	ρ (lb/in^3)	E (psi)	σ_y (psi)	$U_r t$ (lb/in)	EI (lb in^2)	ρA_c (lb/ft)	δ (in)	P_{cr} (lb)
CPVC (sch. 80, 1/2″)	6803K52	1.42	0.84	0.526	0.055	0.36×10^6	7750	26.19	7445.37	0.222	6.27	127.57
PVC (sch. 80, 1/2″)	48855K21	0.52	0.84	0.526	0.051	0.42×10^6	7450	20.75	8686.26	0.206	4.98	148.84
PETG (thick wall)	2044T43	0.5	0.572	0.528	0.046	0.3×10^6	7300	3.908	431.9	0.021	10.2	7.4
Aluminum (6063-T5)	1658T49	0.92	0.625	0.527	0.098	10.4×10^6	15000	1.06	38520.18	0.104	0.57	660.03
Aluminum (2024)	1968T782	10.48	0.625	0.527	0.098	10.4×10^6	42000	8.311	38520.18	0.104	0.57	660.03
Aluminum (6061-T6)	89965K17	3.74	0.625	0.527	0.098	10.4×10^6	35000	5.772	38520.18	0.104	0.57	660.03
Brass (260)	8859K35	4.02	0.563	0.5345	0.308	16×10^6	9000	0.0709	14525.15	0.089	1.29	248.03
Carbon steel[3]	9220K38	2.18	0.625	0.527	0.284	30×10^6	28000	1.281	111115.91	0.302	0.57	1903.94
Copper (122)	50475K23	1.98	0.625	0.527	0.323	17.5×10^6	49400	6.833	64817.62	0.344	1.11	1110.63
Stainless steel (304)	8989K269	4.81	0.625	0.527	0.286	27.5×10^6	40000	2.851	101856.25	0.304	0.63	1745.28

5.8.1 What the table means

Part # refers to the material's product number at McMaster-Carr.

OD is the outer diameter of the tubing.

ID is the inner diameter.

3 Little information about this steel was provided so a weak carbon steel's data was used. This may not be accurate, though, few'll care because steel is a very uncommon barrel material.

ρ is the density of the material.

E is the modulus of elasticity of the material. This is a measure of how easily the material deforms. Higher numbers deform less for a certain load.

σ_y is the yield strength of the material. This is the amount of stress the material can take before permanently deforming.

EI is the modulus of elasticity multiplied by the area moment of inertia of the tube. This describes how stiff the tube is. A stiff tube can not be bent easily. Stiffer tubes are more resistant to bending from their own weight and being struck. They also are resistant to vibration, which does reduce accuracy in firearms but probably has no effect in Nerf.

U_r is called the modulus of resilience. It represents how much energy the material can absorb per unit volume before permanent deformation occurs. Because the modulus of resilience depends solely on the material properties and not the geometry, it is multiplied by the tube thickness so that comparisons of how easily dented the materials are can be made.

To determine if the material will at least yield from being struck, find the kinetic energy density of the material impacting the tube and compare it against the value of $U_r t$.

δ is how much the free end of a horizontal 36 inch barrel will sag in inches calculated with elementary beam theory. The bending is exaggerated with longer barrels, hence why I used a longer barrel here. The variables involved are barrel length, linear density (lb/ft), and stiffness (EI).

P_{cr} is the critical buckling load in pounds for this barrel with a length of 12 inches. Basically, this is how much force must be applied at the end to bend (buckle) the barrel. This really only shows in Nerf when blasters are dropped on their barrels or when barrel tapping.

5.8.2 Each material in summary

CPVC:

- Pros: Reasonably cheap. Excellent impact strength.
- Cons: Heavy.

PVC:

- Pros: Cheap. Excellent impact strength.
- Cons: Heavy.

PETG:

- Pros: Cheap. Lightweight. Clear.
- Cons: Terrible stiffness. Prone to damage. Extremely low critical buckling load.

Al (6063-T5)

- Pros: Cheap. Reasonably lightweight. Very stiff.
- Cons: Will dent easier than other types of aluminum.

Al (2024)

- Pros: Reasonably lightweight. Very stiff. Good impact strength.
- Cons: Expensive.

Al (6061-T6)

- Pros: Reasonably lightweight. Very stiff. Good impact strength.
- Cons: Could be cheaper, but is not expensive.

Brass

- Pros: Can telescope. Reasonably lightweight. Stiff.
- Cons: Very easy to dent. Tarnishes.

Steel

- Pros: Extremely stiff.
- Cons: Extremely heavy. Could be cheaper, but is not expensive.

Copper

- Pros: Extremely stiff. Good impact strength.
- Cons: Extremely heavy. Could be cheaper, but is not expensive.

Stainless steel

- Pros: Extremely stiff. Shiny.
- Cons: Extremely heavy.

5.8.3 Observations

Aluminum and brass are comparable. The primary differences between the two is that aluminum is not designed to telescope like brass, brass can dent far easier than aluminum (shown in the $U_r t$ column), and aluminum is a little more than twice as stiff.

The differences between the different types of aluminum are subtle. They vary only in yield strength and price in this table. The differences in yield strength are amplified in the impact strength before permanent deformation column. Given only minor differences, there is little reason to buy the stronger alloys, at least from McMaster-Carr. Other places (such as Online Metals) may have different prices and stronger alloys may be cheaper and thus worthwhile.

PETG wins huge in weight. However, PETG also loses huge in many strength categories. A thicker walled PETG with the same ID would be desirable. Another clear plastic such as polycarbonate might also be a good idea.

PETG has absolutely terrible stiffness. This is shown quite evidently in the critical buckling load, which is less than 10 pounds. Either be careful with PETG, sleeve it in something else (which would make it much heavier), or use it in a structure like a turret where the geometry would increase the stiffness.

In terms of price, PETG and sch. 80 PVC are winners. 6063 aluminum is the next closest.

The heavier metals, steel, copper, and stainless steel, are rare, and rightly so. They are very strong by virtue of their geometry and material, however, they are also extremely heavy.

5.8.4 Which should I use? / Which is best?

Use materials that are readily available to you and fit the darts you intend to use. For spring blasters you want a tighter fit to allow pressure to build higher due to the slow initial flow. For pneumatics you want a smooth fit because too tight fits introduce non-negligible energy losses to friction, however, be sure that the fit is not loose or energy losses to leaks around the dart will occur.

As all of these materials have approximately the same diameter (until additional tables of different average calibers are made), your decision relies on the properties of the material.

When you want something to telescope, brass is the best choice as it was designed to telescope.

If you want optical clarity and/or low weight, PETG would be a good choice.

If you want a material that mixes a variety of properties, 6063 aluminum is cheap, strong in all measures, and reasonably lightweight.

5.9 Gas reservoirs

5.9.1 General background

Gas reservoirs[4] are systems that supply gas to pneumatic Nerf blasters. There are many varieties, from pressurized gas systems, to "bladders" and systems that expand a liquid into a gas.

If there's anything you take from this section, make it this: *Use gas reservoirs designed to hold gas*. This limits choices to metal LPA and HPA tanks. There are other options but there are good reasons not to use them as I will explain.

Safety notes. Never exceed pressure ratings. Best practice is to use metal tanks at pressures below the maximum operating pressure with a pop-safety valve in place to reduce the likelihood of over-pressurization. HPA tanks already have a pop-safety valve.

Some latex tubes used as bladders will explode when pressurized. The explosion is extremely loud and it will cause hearing damage. Do not use the thickest few of the smallest latex tubes on McMaster-Carr to avoid this problem. Also, do tests with latex tubing with water instead of air — water won't explode in this case.

Regulators. A regulator maintains the outlet pressure at a certain level. Ideally the outlet pressure is completely independent of the inlet pressure, but some regulators are better than others in this respect.

Pressure regulators are necessary for compressed gas stored in a solid tank. Compressed liquid and gas stored in a bladder do not require regulators as the pressure is relatively constant (given constant temperature in liquid systems and minimum expansion in bladder systems), however, if a lower pressure is desired than the vapor pressure of the gas or the bladder pressure of the bladder, a regulator can be used.

There are two types of regulator: relieving and non-relieving. Avoid relieving regulators — they "relieve" the excess pressure by exhausting it. This is very wasteful for Nerf. Also avoid "flow regulators", which do not regulate pressure and merely are ball valves.

For a good small regulator I suggest the Clippard MAR[5] series. These are the smallest (less than $2\frac{1}{2}$ inches long and $\frac{3}{4}$ inches wide) regulators I have encountered. A relieving version exists, so be sure to buy the non-relieving version. Clippard sells them directly, but their shipping is expensive, so I'd suggest looking on eBay. One disadvantage of this regulator is that the outlet pressure is not completely independent of the inlet pressure, so some adjustment is necessary. Dual-regulation (using two regulators in series) is also an option. Two regulators in parallel with both regulators set to the same pressure will increase the flow rate of gas through the regulators compared against one regulator.

4 The title is a little misleading as I include liquid sources like CO_2 on this page as well. Perhaps "propellants" or "gas sources" would be most appropriate.

5 `http://www.clippard.com/store/display.asp?dept_id=163&levels=1`

Other regulators exist and are sold from places like McMaster-Carr, but these regulators are usually designed for air compressors and are very often too large and/or too heavy.

Metal tubing. The ends of thin aluminum tubes available online can be filled with epoxy putty and a pipe fitting, making a lightweight air tank.

Aluminum LPA tanks. Low pressure aluminum cylinders are manufactured by many companies. Catalina Cylinders[6] manufactures a wide variety of size cylinders with capacities that are perfectly appropriate for Nerf.

These LPA (low pressure air, though they can be filled with any non-volatile gas) tanks are very light, not expensive, extremely durable, and very strong. The 85 in^2 tank I received from Catalina Cylinders is extremely safe at 150 psi, weighs less than half a pound, and was very cheap.

Due to the lack of popularity of these tanks, they are only available from the manufacturers directly. Most manufacturers do not sell single units but are willing to help out someone with an odd project. Until someone buys a small stockpile of these tanks to resell, contacting the manufacturer directly is the only way to get one.

Catalina's tanks use a custom fitting called the F1 that connects the uncommon threads of their cylinder to female $\frac{1}{4}$ inch NPT.

- The burst pressure given is just that — the pressure the tank bursts at. A safety factor of 2.5 is common for pressure vessels, so the 600 psi burst cylinders have a pressure rating of 240 psi.
- The gasket on the fitting can burst out at high pressure, leaking gas. Be sure to tighten the cylinder into the fitting adequately. Also, I have successfully used a cable tie tightened around the O-ring to prevent bursts.

High pressure air. HPA tanks are 3000 psi or 4500 psi tanks meant primarily for paintball. These tanks can easily be used for the similar Nerf game.

CO_2.

PET bottles. Many people use soda bottles made from PET plastic as pressure vessels. This is a dangerous practice that I will not suggest, especially given cheap and very attractive alternatives like aluminum LPA tanks.

PET bottles are not designed to hold gas pressure. They are designed to hold liquid under pressure with some gas, and the pressure does not get very high. PET bottles explode very easily, so easily that I never will suggest them. The remainder of my discussion here are reasons not to use PET bottles.

Not all PET bottles are the same. Bottles for non-carbonated beverages such as water often are not designed to hold pressure, even if they appear to be otherwise like carbonated bottles. There also are many differences in bottles between brands. One brand of 2 liter bottle may burst at 100 psi and another may burst at 250 psi. There is the possibility that some bottles are lemons too even in a good brand. The bottles might not stand up to repeated pressurization as they are designed to hold pressure for a certain amount of time.

A "safety factor", that is, the number of times below the failure stress the operating stress is, is very important. Getting useful gas mass from a PET bottle requires pressures near the burst pressure. Most people do not seem to have a problem with this, but it is dangerous. The general safety factor over burst pressure for pressure vessels is 2.5. So if a pressure vessel bursts at 150 psi, you should operate at a maximum of 60 psi to satisfy the safety factor. As the burst pressures vary from brand to brand and even

6 `http://www.catalinacylinders.com/lp.html`

bottle to bottle within brands, no specific recommendation can be made, but I think 60 psi should be safe for most bottles.

Temperature is also something to consider. Pumping gas increases it's temperature and extremely hot temperatures are easily possible (this is called adiabatic heating). Temperatures of 600 F or higher are not uncommon if you neglect heat transfer in your analysis. The high temperatures will be lost relatively quickly to heat transfer. The melting point of PET is 500 SI…of course, the plastic won't reach that temperature due to the specific heat of the plastic[7], but it certainly will be weakened as the heat is transfered. This is an issue soda bottle manufacturers don't worry about because they are not pumping gas into the bottle.

Even operating at safe pressures, I'd be afraid of sudden stresses on the bottle. If you fall on your bottle, will it burst? You could do some tests to figure it out. This is something the water rocket people, who typically pressurize 2 L bottles, don't have to worry about but I think is a very real possibility for Nerf. Spud gunners don't have to worry about this either as they are not playing a game with their creations.

PVC LPA chambers.

Rubber bladders.

Safety. Any pressurized gas container is a bomb if mistreated. Steps can be taken to virtually eliminate the chance of catastrophic failure.

- *Pop-safety and relief valves* release gas when the pressure gets too high. These are available from essentially any place that sells pneumatic components, McMaster-Carr being one of them.
- *Valves to quickly vent chambers* can be particularly useful. Often a gas charge will be left in a tank after a match so they are a good idea to have even not considering safety. Gas chambers, unless specifically designed to be like HPA or CO_2 tanks, should not be left pressurized for extended periods of time because of a phenomena called creep. The rapid discharge of a gas can be loud; an exhaust muffler, available from McMaster-Carr will reduce the noise level greatly
- *"Safety" valves* much in the same vein of mechanical safeties on real guns can be useful to prevent gas flow from the gas source to the gas chamber. Regulators can be adjusted such that no gas can flow through them, making them fill two jobs. Similarly, mechanical stops preventing the motion of the trigger exactly like safeties in real guns can also be employed.

What doesn't improve safety. Wrapping duct tape, similar tapes, denim, or something similar around an inadequate chamber will likely make the chamber less safe. The sleeve will provide no protection in an actual failure. And while it might increase the pressure at which failure occurs, that just increase the amount of potential energy to be released, making the explosion more dangerous.

Wrapping carbon fiber, a second pipe, or something else much stronger than the pressure vessel around the vessel probably will improve safety. However, if you're going to spend the money on those options, you'd be better off turning your attention to proper pressure vessels. Also, the increase in weight could be unacceptable.

5.9.2 Number of shots from a gas source

Questions of the adequacy of a gas reservoir for lasting a Nerf war are common. With some algebra and logic, the number of shots a gas reservoir will provide can be reasonably accurately predicted, greatly aiding the design process.

7 The specific heat is how much energy is required to raise the temperature of a unit cube of material one degree. Basically, two cubes of material at the same temperature but with different specific heats will have different internal energies. Heat transfer is energy transfer, not temperature transfer.

For all of these calculations we'll take the approach that a gas reservoir becomes useless when its output pressure runs below the operating pressure. In some cases more shots can come from a gas reservoir in this state, however, they will be of lower pressure and consequently lower power.

As with most calculations in this section, all pressures are absolute unless noted otherwise.

These calculations are simple and may not be representative of reality. They make a number of assumptions, specifically, that regulators are consistent and their outlet pressure is not dependent on the inlet pressure, that the temperature during the use of a liquid gas source is constant, and that the pressure of a bladder is independent of the bladder's volume.

More complicated and accurate formulas can be derived with more realistic assumptions by similar procedures.

The general approach to find the number of shots one can get from a gas source is to track mass. The conservation of mass is a powerful statement and its use here demonstrates that. In brief, figure out how much mass is removed from the gas reservoir each shot, remove that mass to calculate the new gas reservoir gas mass, and repeat.

Non-regulated gas sources. Non-regulated gas sources are particularly complex; the gas flows from the gas reservoir into the gas chamber until the pressure in the two equalize. This pressure is different for each step; consequently an iterative approach using spreadsheets often is most convenient.

The time scale of the discharge of gas from a gas reservoir in a Nerf war is long, therefore an isothermal assumption is useful. To reiterate points made earlier, isothermal assumptions are good for when there is heat transfer to ensure that the gas temperature is constant — heat transfer is only possible with a rather large time scale.

P_{r0} is the starting pressure of the gas reservoir. V_r is the gas reservoir volume. V_c is the gas chamber volume. P_{atm} is atmospheric pressure. R is the specific gas constant. T is the temperature of the gas. n is the shot number. $m_r(n)$ is the total gas mass in the gas reservoir after shot number n. Similarly, $P_r(n)$ is the pressure of the gas reservoir after shot number n.

Using ideal gas assumptions (specifically, the mass form), note that $m_r(0) = P_{r0}V_r/R/T$ and $P_r(0) = P_{r0}$. Δm_n^{n+1}, the amount of gas mass removed can be found by the ideal gas law as well, but some logic is necessary before finding it. Gas at atmospheric pressure is left in the gas chamber; this mass must be subtracted from the total mass of the gas chamber at the equilibrium pressure (i.e., the common pressure that both the gas reservoir and gas chamber equalize to).

That statement is made more explicit like this:

$$\Delta m_n^{n+1} = \frac{P_{eq} V_r}{RT} - \frac{P_{atm} V_c}{RT} \tag{5.1}$$

What is P_{eq}? It is the pressure the gas reservoir and chamber equalize at (so $P_{eq} = P_r(n+1)$). Using mass conservation and the ideal gas laws we can calculate this. As we'll see, the isothermal assumption will reduce this to Boyle's law.

The gas reservoir has a pressure after the n^{th} shot of $P_r(n)$. The gas chamber is evacuated after a shot so it has a pressure of P_{atm}.

$$m_{\text{r}} + m_{\text{c}} = m_{\text{total}} \tag{5.2}$$

$$m_{\text{r}} = \frac{P_{\text{r}}(n)V_{\text{r}}}{RT} \tag{5.3}$$

$$m_{\text{c}} = \frac{P_{\text{atm}}V_{\text{c}}}{RT} \tag{5.4}$$

$$m_{\text{total}} = \frac{P_{\text{eq}}(V_{\text{c}} + V_{\text{r}})}{RT} \tag{5.5}$$

By combining these equations one arrives at the following:

$$P_{\text{r}}(n)V_{\text{r}} + P_{\text{atm}}V_{\text{c}} = P_{\text{eq}}(V_{\text{c}} + V_{\text{r}}) \tag{5.6}$$

Note that the term RT cancels out from each statement — this is because for an isothermal process PV is proportional to mass — and consequently Boyle's law can be seen as simply a statement of the conservation of mass for isothermal processes.

Solve for P_{eq} and we end up at:

$$P_{\text{eq}} = P_{\text{r}}(n+1) = \frac{P_{\text{r}}(n)V_{\text{r}} + P_{\text{atm}}V_{\text{c}}}{V_{\text{c}} + V_{\text{r}}} \tag{5.7}$$

From here we can derive an explicit formula for Δm_n^{n+1}. Plug P_{eq} into the equation for Δm_n^{n+1}.

$$\Delta m_n^{n+1} = \frac{P_{\text{r}}(n)V_{\text{r}}^2 + P_{\text{atm}}V_{\text{c}}V_{\text{r}}}{RT(V_{\text{c}} + V_{\text{r}})} - \frac{P_{\text{atm}}V_{\text{c}}}{RT} \tag{5.8}$$

The initial conditions of the mass in the gas reservoir (at "shot zero") are:

$$m_{\text{r}}(0) = \frac{P_{\text{r0}}V_{\text{r}}}{RT} \quad , \quad P_{\text{r}}(0) = P_{\text{r0}} \tag{5.9}$$

Unfortunately there does not appear to be any simple equation — even an implicit one — to find the number of shots one can get from a non-regulated gas reservoir. Use of a spreadsheet with the mass difference and the equalization pressure formulas seems ideal. The ugly expressions alone are reason enough to use a regulator; combine that with the consistency of pressure and regulators seem extremely attractive.

To account for the volume of whatever is between the gas reservoir and the gas chamber (tubing, valves, etc.), figure out approximately what the volume that is exhausted and replenished is and add that to the gas chamber volume. The volume that is not exhausted must be added to the gas reservoir volume.

Regulated gas sources. The derivation for regulated gas sources is far cleaner than that of non-regulated sources as the new equalized pressure of each shot does not need to be calculated.

Like with non-regulated sources we'll find Δm, the difference in gas reservoir mass between shots. Each shot is pressurized to P_{reg}, the regulated pressure, and after the gas is exhausted what remains is at atmospheric pressure. Note that here the mass removed per shot is independent of the number of shots taken[8] due to

8 At least until gas reservoir pressure drops below the regulated pressure, but our task is to find where that happens here so we do not worry about that.

the regulator. Following similar logic to that applied for non-regulated sources we find that

$$\Delta m = \frac{V_\mathrm{c}(P_\mathrm{reg} - P_\mathrm{atm})}{RT} \,. \tag{5.10}$$

Here P_reg is the set pressure of the regulator. Using mass conservation (like we did with non-regulated sources again) we can arrive at the following equation:

$$P_\mathrm{r0} V_\mathrm{r} = P_\mathrm{min} V_\mathrm{r} + n V_\mathrm{c}(P_\mathrm{reg} - P_\mathrm{atm}) \tag{5.11}$$

Here P_min is the minimum acceptable pressure of the gas reservoir. Sometimes you want to refill before the gas reservoir's pressure drops below the regulated pressure as a precaution against that — however — most people are interested in how many shots at the regulated pressure they can get from a reservoir, so $P_\mathrm{min} = P_\mathrm{reg}$. Rewriting the equation we arrive at:

$$n = \frac{V_\mathrm{r}(P_\mathrm{r0} - P_\mathrm{reg})}{V_\mathrm{c}(P_\mathrm{reg} - P_\mathrm{atm})} \tag{5.12}$$

Round this result down as you can't get a fraction of a shot.

Note that this assumes the regulator is perfect and regulates at the pressure it was set at consistently. Some regulators are better than others at this; the small Clippard MAR regulators I have used have an outlet pressure that seems rather dependent on the inlet pressure. This I do not consider to be a major problem as adjustments can be easily made on the fly.

Non-regulated liquid sources. Examples of liquid sources are CO_2 and propane.

Regulated liquid sources.

Bladders. For a bladder, you can assume that the pressure is approximately constant and basically divide the total mass a bladder contains by the total mass used by a shot (calculated via the ideal gas law) to calculate how many shots the tank has. For a more complicated model, assume that the pressure drops linearly as a function of volume from the bladder's operating pressure to about 25% less than that when the bladder is empty. Real bladders' pressure decreases by approximately that much, though not linearly.

5.9.3 Latex tubing

More information about latex tubing is available online [Tre10].

Pressure generated. The pressure rating places like McMaster-Carr tells is not a "maximum pressure" rating. Latex tubing keeps pressure approximately constant. If air of higher pressure is pumped in, it will expand until it reaches the equilibrium pressure of the tube. Latex tubing does not break from pressures that are too high — it breaks when too much volume is put in. This is a very common misconception.

This pressure rating is approximately the pressure needed for the tube to expand at first. Pressure spikes as a function of volume for the first few pumps, and once the rubber expands, the pressure is approximately independent of volume. The actual operating pressure of the tubing is approximately two-thirds of the reported pressure on McMaster-Carr. This figure is based on tests of a single tube.

An approximate equation for pressure based on data provided by McMaster-Carr is

$$P(t, d_\mathrm{i}) = 31.4\sqrt{t} + \frac{18.0t}{d_\mathrm{i}} \tag{5.13}$$

where

P is the pressure in psi,

d_i is the internal diameter of the tube (unfilled) in in, and

t is the tube wall thickness in in.

Approximate expansion diameter. I'll start with a few assumptions. When the tube expands, it is assumed to stop expanding when its expanded inner diameter equals a certain multiple of the unexpanded diameter. This is justifiable based on the observed behavior of the rubber. The Gent hyperelastic model makes the same assumption. My second assumption is that when the tube is fully expanded the cross-sectional area of the expanded tube is a multiple of the cross-sectional area of the unexpanded tube. I originally anticipated these areas would be equal as I know for small deformations, rubber is approximately incompressible (i.e., the volume is preserved). But that's only applicable for volume, not area; the tube is expanding in length too. Also, it is only applicable for small deformations. I do not know how the rubber will act for large deformations.

D_o is the outer diameter of the tube. D_i is the inner diameter of the tube. t is the tube wall thickness. The superscript e refers to the expanded state. A variable or constant without e is in the unexpanded state or it does not refer to any state (like the constants).

$A = \frac{\pi}{4}(D_\mathrm{o}^2 - D_\mathrm{i}^2)$ is the equation for the cross-sectional area of the unexpanded tube.

$A^\mathrm{e} = \frac{\pi}{4}[(D_\mathrm{o}^\mathrm{e})^2 - (D_\mathrm{i}^\mathrm{e})^2]$ is the equation for the cross-sectional area of the expanded tube.

$D_\mathrm{i}^\mathrm{e} = C_1 D_\mathrm{i}$ is my assumption about when the tubes stop expanding.

$A^\mathrm{e} = C_2 A$, where C_2 is my assumption about the cross-sectional areas of the tubes when they stop expanding.

Plugging all these equations together and solving for D_o^e, I find that $D_\mathrm{o}^\mathrm{e} = \sqrt{C_1^2 D_\mathrm{i}^2 + 4tC_2(D_\mathrm{i} + t_\mathrm{i})}$.

The data points I have available:

- Unexpanded ID: 3/8 in, wall thickness: 3/16 in, expanded OD: a bit more than 3 in (from memory)
- Unexpanded ID: 1/8 in, wall thickness: 3/16 in, expanded OD: 1.25 in
- Unexpanded ID: 1/8 in, wall thickness: 7/32 in, expanded OD: 1.375 in

A linear regression leads to $C_1 = 7.35$ and $C_2 = 3.33$ (R^2 = 0.9999, which would definitely be lower if there were more data points). These constants seem reasonable given my understanding of the phenomena, so it is reasonable to accept that the tubes' inner diameters expand to about 7.35 times their original inner diameter and the cross-sectional area increases to 3.33 times the original cross-sectional area.

The equation above with these constants can be used to find the expanded outer diameter. The equation with C_1 can find the expanded inner diameter. The definition $D_\mathrm{o}^\mathrm{e} = D_\mathrm{i}^\mathrm{e} + 2t^\mathrm{e}$ can find the expanded wall thickness.

This formula is the best I can do with the data I have on hand.

An equation for the inner diameter of a tube if the outer diameter is restricted (i.e., does not fully expand) follows. This is based on the assumption that the area ratio scales linearly with the inner diameter ratio.

$$D_\mathrm{i}^\mathrm{e} = (C_1 D_\mathrm{i})^{-1}[\sqrt{4(tC_2)^2(D_\mathrm{i} + t)^2 + (C_1 D_\mathrm{i})^2 (D_\mathrm{o}^\mathrm{e})^2} - 2tC_2(D_\mathrm{i} + t)] \tag{5.14}$$

5.9.4 Stresses in gas reservoirs

5.10 Springs

5.11 Plunger rods

Desired performance characteristics from a plunger rod.

Available materials and material property data.

Static analysis of forces on plunger rods.

Dynamic analysis of forces on plunger rods.

Conclusions.

5.12 Common failures

5.13 Material properties

6 Testing and test data

6.1 Range

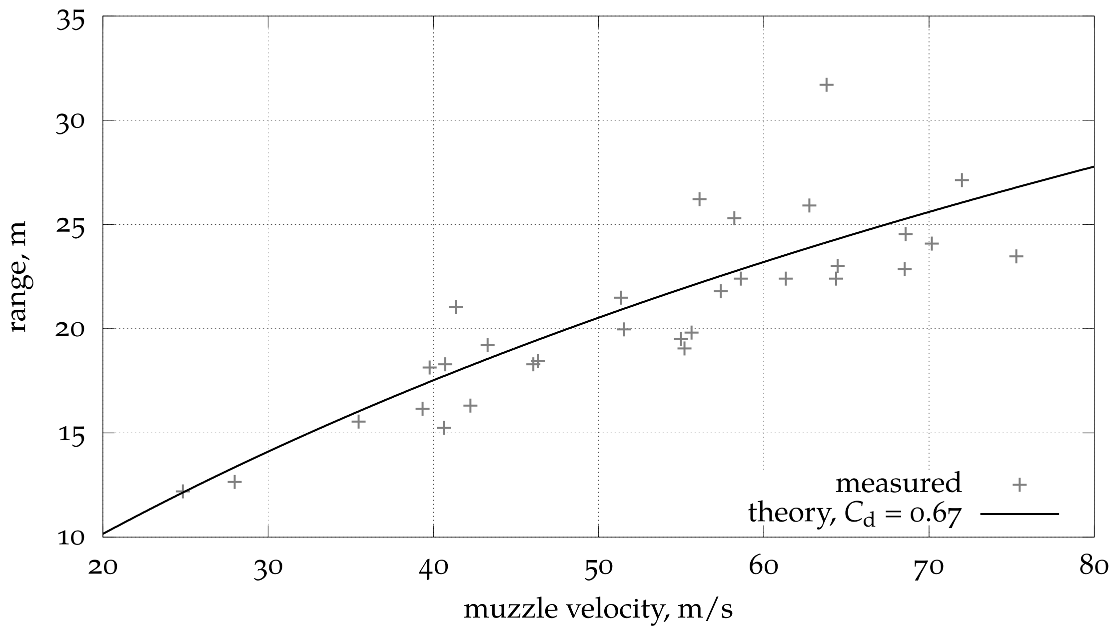

Figure 12: Measured ranges as a function of muzzle velocity vs. a theoretical model (eqn. 3.61) with $C_d = 0.67$. Experimental data from Beaver [Bea12].

6.2 Muzzle velocity

6.2.1 Compensating chronometer readings to account for drag

If a chronometer is used to measure muzzle velocity, some compensation to account for the distance between the barrel tip (L_c) and the distance between the chronometer sensors (L_s) might be necessary. A

Table 2: Material properties

This data represents only generic figures found in textbooks and online reference. Use data provided by manufacturers and suppliers or ask for such data for the most precise and accurate numbers.

Material	**Elastic modulus, E**		**Density, ρ**		**Ultimate strength, σ_u**		**Yield strength, σ_y**	
	Mpsi	GPa	lb/in^3	g/cm^3	ksi	MPa	ksi	MPa
Plastics								
ABS	0.31	2.1	0.0376	1.04	—	—	6.1	42
Acrylic	0.43	2.9	0.0426	1.18	10.2	70	9.6	66
Delrin	0.35	2.4	0.0513	1.42	9.0	62	—	—
Nylon 6/6	0.4	2.8	0.0412	1.14	11	75	6.5	45
Polycarbonate	0.35	2.4	0.0433	1.20	9.5	65	9	62
Polystyrene	0.45	3.1	0.0374	1.03	8	55	8.0	55
PET	0.4	2.8	0.0494	1.37	6.6	46	—	—
PETG	0.3	2.1	0.0459	1.27	7.7	53	7.3	50
LDPE	0.042	0.29	0.0332	0.919	1.9	13	1.4	9.7
UHMW-PE	0.12	0.83	0.0336	0.0929	5.8	40	3.1	21
HDPE	0.23	1.6	.0347	0.959	—	—	4.2	29
PVC	0.41	2.8	0.0513	1.42	—	—	7.5	52
CPVC	0.36	2.5	0.0531	1.47	—	—	7.3	50
Metals								
Al (2024-T3)	10.4	72	0.100	2.73	70	483	50	345
Al (6061-T6)	10.4	72	0.098	2.70	45	310	40	276
Al (6063-T5)	10.4	72	0.098	2.70	27	186	21	145
Brass (260)	16	110	0.308	8.53	61.6	425	52.2	360

chronometer really just measures Δt and uses $\Delta t \equiv L_s/\overline{V}$ to find $\overline{V}$, the apparent muzzle velocity. Using

$$x(t) = \frac{\log(V_0 K t + 1)}{K} \, . \tag{3.25}$$

we can compensate for these effects. If the equation above is rearranged for t and the time the projectile takes to reach the first chronometer sensor (t_1) is subtracted from the time the projectile takes to reach the second chronometer sensor (t_2), we can define $\Delta t \equiv t_2 - t_1$ and find

$$V_0 = \overline{V} \left(\frac{e^{K(L_c - L_s)} - e^{K L_c}}{L_s K} \right) \, . \tag{6.1}$$

Generally $V_0/\overline{V}$ is near 1. For example, for a typical Nerf dart with $L_c = 5\,\text{cm}$ and $L_s = 10\,\text{cm}$, $V_0/\overline{V} = 1.0046$.

6.3 Dart mass

6.4 Dart stability

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